Combining Algebra and Geometry

Analytic geometry, which combines algebra and geometry, provides a tremendously powerful tool for visualizing equations with two variables.

We begin this chapter with a description of the coordinate plane, including a discussion of distance and length. Then we turn our attention to lines and their slopes, which are simple concepts that have immense importance.

We then investigate quadratic expressions. We will see how to complete the square and solve quadratic equations. Quadratic expressions will also lead us to the conic sections (ellipses, parabolas, and hyperbolas).

We conclude the chapter by learning methods for computing the area of triangles, trapezoids, circles, and ellipses.
2.1 The Coordinate Plane

LEARNING OBJECTIVES
By the end of this section you should be able to
- locate points in the coordinate plane;
- graph equations with two variables in the coordinate plane, possibly using technology;
- compute the distance between two points;
- compute the circumference of a circle.

Coordinates
Recall how the real line is constructed: We start with a horizontal line, pick a point on it that we label 0, pick a point to the right of 0 that we label 1, and then we label other points using the scale determined by 0 and 1 (see Section 1.1 to review the construction of the real line).

The coordinate plane is constructed in a similar fashion, but using a horizontal and a vertical line rather than just a horizontal line.

The coordinate plane
- The coordinate plane is constructed by starting with a horizontal line and a vertical line in a plane. These lines are called the coordinate axes.
- The intersection point of the coordinate axes is called the origin; it receives a label of 0 on both axes.
- On the horizontal axis, pick a point to the right of the origin and label it 1. Then label other points on the horizontal axis using the scale determined by the origin and 1.
- Similarly, on the vertical axis, pick a point above the origin and label it 1. Then label other points on the vertical axis using the scale determined by the origin and 1.

Sometimes it is important to use the same scale on both axes, as is done in the figure above. Other times it may be more convenient to use two different scales on the two axes.

A point in the plane is identified with its coordinates, which are written as an ordered pair of numbers surrounded by parentheses, as described below.
**Coordinates**

- The first coordinate indicates the horizontal distance from the origin, with positive numbers corresponding to points right of the origin and negative numbers corresponding to points left of the origin.

- The second coordinate indicates the vertical distance from the origin, with positive numbers corresponding to points above the origin and negative numbers corresponding to points below the origin.

**EXAMPLE 1**

Locate on a coordinate plane the following points:
(a) \((2, 1)\);  
(b) \((-1, 2.5)\);  
(c) \((-2.5, -2.5)\);  
(d) \((3, -2)\).

**SOLUTION**

(a) The point \((2, 1)\) can be located by starting at the origin, moving 2 units to the right along the horizontal axis, and then moving up 1 unit; see the figure below.

(b) The point \((-1, 2.5)\) can be located by starting at the origin, moving 1 unit to the left along the horizontal axis, and then moving up 2.5 units; see the figure below.

(c) The point \((-2.5, -2.5)\) can be located by starting at the origin, moving 2.5 units to the left along the horizontal axis, and then moving down 2.5 units; see the figure below.

(d) The point \((3, -2)\) can be located by starting at the origin, moving 3 units to the right along the horizontal axis, and then moving down 2 units; see the figure below.

The notation \((-1, 2.5)\) could denote either the point with coordinates \((-1, 2.5)\) or the open interval \((-1, 2.5)\). You should be able to tell from the context which meaning is intended.

These coordinates are sometimes called **rectangular coordinates** because each point’s coordinates are determined by a rectangle, as shown in this figure.

The horizontal axis is often called the **x-axis** and the vertical axis is often called the **y-axis**. In this case, the coordinate plane can be called the **xy-plane**. However, other variables can also be used, depending on the problem at hand.
Regardless of the names of the axes, remember that the first coordinate corresponds to horizontal distance from the origin and the second coordinate corresponds to vertical distance from the origin.

If the horizontal axis has been labeled the \( x \)-axis, then the first coordinate of a point is often called the \textbf{x-coordinate}. Similarly, if the vertical axis has been labeled the \( y \)-axis, then the second coordinate is often called the \textbf{y-coordinate}.

The potential confusion of this terminology becomes apparent when we want to consider a point whose coordinates are \((y,x)\); here \( y \) is the \( x \)-coordinate and \( x \) is the \( y \)-coordinate. Furthermore, always calling the first coordinate the \( x \)-coordinate will lead to confusion when the horizontal axis is labeled with another variable such as \( t \) or \( \theta \).

**Graphs of Equations**

The coordinate plane allows us to visualize the set of points satisfied by equations with two variables.

**Graph of an equation**

The graph of an equation with two variables is the set of points in the corresponding coordinate plane satisfied by the equation.

**Example 2**

The graph of the equation \( 4x^4 + y^2 = 2 \) is shown in the margin.

(a) Where does this graph intersect the \( x \)-axis?

(b) Where does this graph intersect the \( y \)-axis?

(c) Is the point \((\frac{1}{2}, \frac{2}{3})\) on this graph?

(d) Find four points on this graph that are not on either of the coordinate axes.

**Solution**

(a) The \( x \)-axis is the set of points where \( y = 0 \). Thus to find the points where the graph intersects the \( x \)-axis, substitute \( y = 0 \) in the equation \( 4x^4 + y^2 = 2 \) to obtain the equation \( 4x^4 = 2 \), which implies that \( x^4 = \frac{1}{2} \), which implies that \( x = 1/2^{1/4} \approx 0.8 \) or \( x = -1/2^{1/4} \approx -0.8 \) (we will discuss fractional exponents in detail in Section 5.1, but you are probably already familiar with this concept from a previous course).

Thus the points where the graph intersects the \( x \)-axis are approximately \((0.8,0)\) and \((-0.8,0)\). Make sure you can locate these points on the graph in the margin.

(b) The \( y \)-axis is the set of points where \( x = 0 \). Thus to find the points where the graph intersects the \( y \)-axis, substitute \( x = 0 \) in the equation \( 4x^4 + y^2 = 2 \) to obtain the equation \( y^2 = 2 \), which implies that \( y = \sqrt{2} \approx 1.4 \) or \( y = -\sqrt{2} \approx -1.4 \).

Thus the points where the graph intersects the \( y \)-axis are approximately \((0,1.4)\) and \((0,-1.4)\). Make sure you can locate these points on the graph in the margin.
(c) Asking whether \((\frac{1}{2}, \frac{7}{5})\) is on the graph of the equation \(4x^4 + y^2 = 2\) is equivalent to asking whether \(4\left(\frac{1}{2}\right)^4 + \left(\frac{7}{5}\right)^2\) equals 2. A little arithmetic shows that \(4\left(\frac{1}{2}\right)^4 + \left(\frac{7}{5}\right)^2\) equals \(\frac{221}{125}\), which equals 2.21, which is close but not equal to 2. Thus the point \((\frac{1}{2}, \frac{7}{5})\) is not on the graph of \(4x^4 + y^2 = 2\).

The red dot in the figure in the margin shows the point \((\frac{1}{2}, \frac{7}{5})\). As you can see, it is not on the graph of \(4x^4 + y^2 = 2\), although it is close. The vertical line segment has endpoints at \((\frac{1}{2}, 0)\) and \((\frac{1}{2}, \frac{7}{5})\), showing that the red dot has first coordinate equal to \(\frac{1}{2}\).

(d) To find some points on the graph not on the coordinate axes, substitute \(y = 1\) in the equation \(4x^4 + y^2 = 2\) to obtain the equation \(4x^4 = 1\), which implies that \(x = 1/4^{1/4} \approx 0.7\) or \(x = -1/4^{1/4} \approx -0.7\). Thus \((1/4^{1/4}, 1)\) and \((-1/4^{1/4}, 1)\), which are approximately \((0.7, 1)\) and \((-0.7, 1)\), are points on the graph.

Because \((-y)^2 = y^2\), the points \((1/4^{1/4}, -1)\) and \((-1/4^{1/4}, -1)\), which are approximately \((0.7, -1)\) and \((-0.7, -1)\), are also on the graph.

The blue dots in the figure in the margin show the four points on the graph that we have just found. The top horizontal line segment has endpoints at \((-1/4^{1/4}, 1)\) and \((1/4^{1/4}, 1)\), showing that these two points have second coordinate equal to 1. The bottom horizontal line segment has endpoints at \((-1/4^{1/4}, -1)\) and \((1/4^{1/4}, -1)\), showing that these two points have second coordinate equal to \(-1\).

Later in this chapter we will discuss in detail graphs that are lines, circles, ellipses, parabolas, and hyperbolas. Throughout the book we will discuss the graphs of a large variety of functions that we will study. Meanwhile, use Wolfram\slash Alpha or a graphing calculator to experiment with the graphs of equations.

The next example uses Wolfram\slash Alpha in the solution, but you could use a graphing calculator or any other technology instead.

(a) Graph the equation \(2t^3 + z = z^3 + t\).
(b) Graph the equation \(2t^3 + z = z^3 + t\) for \(t\) in the interval \([-6, 6]\).
(c) Determine whether or not the point \((4.5, 5.5)\) is on the graph of \(2t^3 + z = z^3 + t\).

**SOLUTION**

(a) Point a web browser to www.wolframalpha.com. In the one-line entry box that appears near the top of the web page, enter

```
graph 2t^3 + z = z^3 + t
```

and then press the enter key on your keyboard or click the = box on the right side of the Wolfram\slash Alpha entry box, getting a graph that should look like the one shown in the margin here.

Wolfram\slash Alpha selects the variable lowest in alphabetical order to use as the horizontal axis and the variable highest in alphabetical order to use as the vertical axis, as has been done here with \(t\) and \(z\) and as is usually what you want to do when the variables are \(x\) and \(y\).
(b) Unlike the previous graph we examined, the graph of \(2t^3 + z = z^3 + t\) includes points arbitrarily far left, right, up, and down. Thus the figure shown in the margin above is not the entire graph of \(2t^3 + z = z^3 + t\). No figure could show the entire graph of this equation.

Wolfram|Alpha usually chooses an interesting part of the graph to show, but you may want to zoom in to see more detail or zoom out to see a bigger picture. For example, to zoom out to see the graph for \(t\) in the interval \([-6, 6]\), type

\[
\text{graph } 2t^3 + z = z^3 + t \text{ from } t = -6 \text{ to } 6
\]
in a Wolfram|Alpha entry box to obtain the graph shown below.

\[
\text{The graph of } 2t^3 + z = z^3 + t \text{ for } t \text{ in the interval } [-6, 6].
\]

(c) From the graph above, it appears possible that the point \((4.5, 5.5)\) may be on the graph of \(2t^3 + z = z^3 + t\). To check whether or not this is true, we substitute \(t = 4.5\) and \(z = 5.5\) into the equation and check whether we get a true statement. To do this check using Wolfram|Alpha, type

\[
\text{is } 2 * 4.5^3 + 5.5 = 5.5^3 + 4.5
\]
in a Wolfram|Alpha entry box to obtain the answer False. Thus the point \((4.5, 5.5)\) is not on the graph of \(2t^3 + z = z^3 + t\).

\[\]
Here is another example, this time with neither of the points being the origin.

Find the distance between the points (5, 6) and (2, 1).

**SOLUTION**  The distance between the points (5, 6) and (2, 1) is the length of the hypotenuse in the right triangle shown here. The horizontal side of this triangle has length $5 - 2$, which equals 3, and the vertical side of this triangle has length $6 - 1$, which equals 5. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{3^2 + 5^2}$, which equals $\sqrt{34}$.

More generally, to find the formula for the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$, consider the right triangle in the figure below:

![Figure](image.png)

*The length of the hypotenuse equals the distance between $(x_1, y_1)$ and $(x_2, y_2)$.*

Starting with the points $(x_1, y_1)$ and $(x_2, y_2)$ in the figure above, make sure you understand why the third point in the triangle (the vertex at the right angle) has coordinates $(x_2, y_1)$. Also, verify that the horizontal side of the triangle has length $x_2 - x_1$ and the vertical side of the triangle has length $y_2 - y_1$, as indicated in the figure above. The Pythagorean Theorem then gives the length of the hypotenuse, leading to the following formula:

**Distance between two points**

The distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$  

Using the formula above, we can now find the distance between two points without drawing a figure.

Find the distance between the points (3, 1) and (−4, −99).

**SOLUTION**  The distance between these two points is $\sqrt{(3 - (-4))^2 + (1 - (-99))^2}$, which equals $\sqrt{7^2 + 100^2}$, which equals $\sqrt{49 + 10000}$, which equals $\sqrt{10049}$.  

As a special case of this formula, the distance between a point $(x, y)$ and the origin is $\sqrt{x^2 + y^2}$.  

Length, Perimeter, and Circumference

The length of a line segment is the distance between the two endpoints. For example, the length of the line segment connecting the points \((-1, 4)\) and \((2, 6)\) equals

\[
\sqrt{(2 - (-1))^2 + (6 - 4)^2},
\]

which equals \(\sqrt{13}\).

Defining the length of a path or curve in the coordinate plane is more complicated. A rigorous definition requires calculus, so we use the following intuitive definition:

**Length**

The length of a path or curve can be determined by placing a string on the path or curve and then measuring the length of the string when it is straightened into a line segment.

**Example 7**

Find the length of the path shown here consisting of the line segment connecting \((-2, 2)\) with \((5, 3)\) followed by the line segment connecting \((5, 3)\) with \((2, 1)\).

**Solution**

The first line segment has length

\[
\sqrt{(5 - (-2))^2 + (3 - 2)^2},
\]

which equals \(\sqrt{50}\). The second line segment has length

\[
\sqrt{(2 - 5)^2 + (1 - 3)^2},
\]

which equals \(\sqrt{13}\). Thus this path has length \(\sqrt{50} + \sqrt{13}\).

You are probably already familiar with two other words that are used to denote the lengths of certain paths that begin and end at the same point. One word probably would have been enough, but the two following words are commonly used:

**Perimeter and circumference**

- The perimeter of a polygon is the length of the path that surrounds the polygon.
- The circumference of a region is the length of the curve that surrounds the region.
(a) What is the perimeter of an equilateral triangle with sides of length $s$?
(b) What is the perimeter of a square with sides of length $s$?
(c) What is the perimeter of a rectangle with width $w$ and height $h$?

**SOLUTION**

(a) An equilateral triangle with sides of length $s$ has perimeter $3s$.
(b) A square with sides of length $s$ has perimeter $4s$.
(c) A rectangle with width $w$ and height $h$ has perimeter $2w + 2h$.

The perimeter of an equilateral triangle is proportional to the length of one of its sides (the ratio is 3) and the perimeter of a square is proportional to the length of one of its sides (the ratio is 4). Thus it is reasonable to believe that the circumference of a circle is proportional to its diameter.

Physical experiments confirm this belief. For example, suppose you have a very accurate ruler that can measure lengths with 0.01-inch accuracy. If you place a string on top of a circle with diameter 1 inch, then straighten the string to a line segment, you will find that the string has length about 3.14 inches. Similarly, if you place a string on top of a circle with diameter 2 inches, then straighten the string to a line segment, you will find that the string has length about 6.28 inches. Thus the circumference of a circle with diameter two inches is twice the circumference of a circle with diameter 1 inch.

![Circle Diagram](https://example.com/circle_diagram.png)

The circle on the left has been straightened into the line segment on the right. A measurement shows that this line segment is approximately 3.14 times as long as the diameter of the circle, which is shown above in red.

Similarly, you will find that for any circle you measure, the ratio of the circumference to the diameter is approximately 3.14. The exact value of this ratio is so important that it gets its own symbol:

\[ \pi \]

The ratio of the circumference to the diameter of a circle is called \( \pi \).

It turns out that \( \pi \) is an irrational number. For most practical purposes, 3.14 is a good approximation of \( \pi \)—the error is about 0.05%. If more accurate computations are needed, then 3.1416 is an even better approximation—the error is about 0.0002%.

Just for fun, here are the first 504 digits of \( \pi \):

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986628034825342117067982148086513282306647093846095505822317253594081284811174502841027019385211055596446229893494002338735520172024915161559724245286050360859519361332228467034967802075367432144155413065354079449539574613198246280726659145743887472539925506878817203517085841111524401572932897071426271332360555311424220814067159142330564391785008765270162560057643358300220513153741756241432151457323168468180582626473930512800027163672545510579202513427674193805847375450381383984379946277638668425676800526894225461423919699478647265217803374674250891127875902803523274762851877499877051364673705685360300119452807504546982435345692356161208791332078224119595377704394157701429388986010891221311473292873016436567911768512053955484013846223078622307217422778866820703871523194171693001351248512848073056585258531715676217566623459795916152682129132279048634671219561425917218258967539...
A fraction that approximates \( \pi \) well is \( \frac{22}{7} \) (notice how page 22 is numbered in this book)—the error is about 0.04%. A fraction that approximates \( \pi \) even better is \( \frac{355}{113} \)—the error is extremely small at about 0.000008%.

Keep in mind that \( \pi \) is not equal to 3.14 or 3.1416 or \( \frac{22}{7} \) or \( \frac{355}{113} \). All of these are useful approximations, but \( \pi \) is an irrational number that cannot be represented exactly as a decimal number or as a fraction.

We have defined \( \pi \) to be the number such that a circle with diameter \( d \) has circumference \( \pi d \). Because the diameter of a circle is equal to twice the radius, we have the following formula:

**Circumference of a circle**

A circle with radius \( r \) has circumference \( 2\pi r \).

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**EXAMPLE 9**

Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal the total length of the straight part of the track. What dimensions should the track have?

**SOLUTION**

We want the total length of the straight part of the track to be 200 meters. Thus each of the two straight pieces must be 100 meters long. Hence we take \( c = 100 \) meters in the figure in the margin.

We also want the total length of the two half-circles to be 200 meters. Thus we want \( \pi d = 200 \). Hence we take \( d = \frac{200}{\pi} \approx 63.66 \) meters in the figure in the margin.

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**EXERCISES**

For Exercises 1–8, give the coordinates of the specified point using the figure below:

9. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): \( (1, 2), (-2, 2), (-3, -1), (2, -3) \).

10. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): \( (2.5, 1), (-1, 3), (-1.5, -1.5), (1, -3) \).

11. Find the distance between the points \( (3, -2) \) and \( (-1, 4) \).

12. Find the distance between the points \( (-4, -7) \) and \( (-8, -5) \).

13. Find two choices for \( t \) such that the distance between \( (2, -1) \) and \( (t, 3) \) equals 7.

14. Find two choices for \( t \) such that the distance between \( (3, -2) \) and \( (1, t) \) equals 5.
15. Find two choices for $b$ such that $(4, b)$ has distance 5 from $(3, 6)$.
16. Find two choices for $b$ such that $(b, -1)$ has distance 4 from $(3, 2)$.
17. Find two points on the horizontal axis whose distance from $(3, 2)$ equals 7.
18. Find two points on the horizontal axis whose distance from $(1, 4)$ equals 6.
19. A ship sails north for 2 miles and then west for 5 miles. How far is the ship from its starting point?
20. A ship sails east for 7 miles and then south for 3 miles. How far is the ship from its starting point?

Use the following information concerning Manhattan (New York City) for Exercises 21–22:

- **Avenues in Manhattan run roughly north-south; streets run east-west.**
- **For much of its length, Broadway runs diagonally across the grid formed by avenues and streets.**
- **The distance between consecutive avenues in Manhattan is 922 feet.**
- **The distance between consecutive streets in Manhattan is 260 feet.**

21. Suppose you walk from the corner of Central Park at 8th Avenue and 59th Street to 10th Avenue and 71st Street.
   (a) What is the length of your path if you walk along 8th Avenue to 71st Street, then along 71st Street to 10th Avenue?
   (b) What is the length of your path if you walk along Broadway, which goes in a straight line from 8th Avenue and 59th Street to 10th Avenue and 71st Street?
   (c) How much shorter is the direct path along Broadway than walking along 8th Avenue and 71st Street?
   (d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 8th Avenue and 71st Street?
22. Suppose you walk from 5th Avenue and 24th Street to Times Square at 7th Avenue and 42nd Street.
   (a) What is the length of your path if you walk along 5th Avenue to 42nd Street, then along 42nd Street to 7th Avenue?
   (b) What is the length of your path if you walk along Broadway, which goes in a straight line from 5th Avenue and 24th Street to 7th Avenue and 42nd Street?
   (c) How much shorter is the direct path along Broadway than walking along 5th Avenue and 42nd Street?
   (d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 5th Avenue and 42nd Street?
23. Find two points on the vertical axis whose distance from $(5, -1)$ equals 8.
24. Find two points on the vertical axis whose distance from $(2, -4)$ equals 5.
25. Find the perimeter of the triangle that has vertices at $(1, 2), (5, -3),$ and $(-4, -1)$.
26. Find the perimeter of the triangle that has vertices at $(-3, 1), (4, -2),$ and $(5, -1)$.
27. Find the radius of a circle that has circumference 12 inches.
28. Find the radius of a circle that has circumference 20 feet.
29. Find the radius of a circle that has circumference 8 more than its diameter.
30. Find the radius of a circle that has circumference 12 more than its diameter.
31. Find an equation whose graph in the $xy$-plane is the set of points whose distance from the origin is 3.
32. Find an equation whose graph in the $xy$-plane is the set of points whose distance from the origin is 5.
33. Find the length of the graph of the equation $x^2 + y^2 = 9$.
34. Find the length of the graph of the equation $x^2 + y^2 = 25$. 
35. Find an equation whose graph in the $rt$-plane is the set of points whose distance from $(3, 2)$ is 2.

36. Find an equation whose graph in the $bc$-plane is the set of points whose distance from $(-2, 1)$ is 3.

37. Find two points that have distance 2 from the origin and distance 3 from $(3,0)$.

38. Find two points that have distance 3 from the origin and distance 2 from $(2,0)$.

39. Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal half the total length of the straight part of the track. What dimensions should the track have?

40. Suppose you want to design a 200-meter indoor track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal three-fourths the total length of the straight part of the track. What dimensions should the track have?

41. Suppose a rope is just long enough to cover the equator of the Earth. About how much longer would the rope need to be so that it could be suspended seven feet above the entire equator?

42. Suppose a satellite is in orbit one hundred miles above the equator of the Earth. About how much further does the satellite travel in one orbit than would a person traveling once around the equator on the surface of the Earth?

PROBLEMS

Some problems require considerably more thought than the exercises. Unlike exercises, problems often have more than one correct answer.

43. Find two points, one on the horizontal axis and one on the vertical axis, such that the distance between these two points equals 15.

44. Explain why there does not exist a point on the horizontal axis whose distance from $(5,4)$ equals 3.

45. Use Wolfram|Alpha or a calculator to find the distance between the points $(-21, -15)$ and $(17, 28)$. [In Wolfram|Alpha, you can do this by typing distance from (-21, -15) to (17, 28) in an entry box. Note that in addition to the distance in both exact and approximate form, you get a figure showing the two points. Experiment with finding the distance between other pairs of points, and notice the placement of the points on the figures produced by Wolfram|Alpha.]

46. Find six distinct points whose distance from the origin equals 3.

47. Find six distinct points whose distance from $(3,1)$ equals 4.

48. Graph the equation $x^4 + y^4 = 1$.

49. (a) Graph the equation $y^3 - 3y = x^2$ for $x$ in the interval $[-3, 3]$.

(b) What is unusual about this graph as compared to other graphs we have examined?

50. Show that a square whose diagonal has length $d$ has perimeter $2\sqrt{2}d$.

51. The figure below illustrates an isosceles right triangle with legs of length 1, along with one-fourth of a circle centered at the right-angle vertex of the triangle. Using the result that the shortest path between two points is a line segment, explain why this figure shows that $2\sqrt{2} < \pi$. 

![Diagram of an isosceles right triangle with one-fourth of a circle on one side.](image)
WORKED-OUT SOLUTIONS to Odd-numbered Exercises

Do not read these worked-out solutions before first attempting to do the exercises yourself. Otherwise you may merely mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises (even if they have not been assigned) and check your answers here. If you get stuck on an exercise, reread the section of the textbook—then try the exercise again. If you are still stuck, then look at the worked-out solution here.

For Exercises 1–8, give the coordinates of the specified point using the figure below:

1. A

**SOLUTION** To get to the point A starting at the origin, we must move 3 units right and 2 units up. Thus A has coordinates (3, 2).

Numbers obtained from a figure should be considered approximations. Thus the actual coordinates of A might be (3.01, 1.98).

3. C

**SOLUTION** To get to the point C starting at the origin, we must move 1 unit left and 2 units up. Thus C has coordinates (−1, 2).

5. E

**SOLUTION** To get to the point E starting at the origin, we must move 3 units left and 2 units down. Thus E has coordinates (−3, −2).

7. G

**SOLUTION** To get to the point G starting at the origin, we must move 1 unit right and 2 units down. Thus G has coordinates (1, −2).

9. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): (1, 2), (−2, 2), (−3, −1), (2, −3).

**SOLUTION**

11. Find the distance between the points (3, −2) and (−1, 4).

**SOLUTION** The distance between the points (3, −2) and (−1, 4) equals \( \sqrt{(-1 - 3)^2 + (4 - (-2))^2} \), which equals \( \sqrt{(-4)^2 + 6^2} \), which equals \( \sqrt{16 + 36} \), which equals \( \sqrt{52} \), which can be simplified as follows:

\[ \sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}. \]

Thus the distance between the points (3, −2) and (−1, 4) equals \( 2\sqrt{13} \).

13. Find two choices for \( t \) such that the distance between (2, −1) and (\( t, 3 \)) equals 7.

**SOLUTION** The distance between (2, −1) and (\( t, 3 \)) equals \( \sqrt{(t - 2)^2 + 16} \).
We want this to equal 7, which means that we must have
\[(t - 2)^2 + 16 = 49.\]
Subtracting 16 from both sides of the equation above gives
\[(t - 2)^2 = 33,\]
which implies that \(t - 2 = \pm \sqrt{33}.\) Thus \(t = 2 + \sqrt{33}\) or \(t = 2 - \sqrt{33}.\)

15. Find two choices for \(b\) such that \((4, b)\) has distance 5 from \((3, 6).\)

**SOLUTION** The distance between \((4, b)\) and \((3, 6)\) equals
\[\sqrt{1 + (6 - b)^2}.\]
We want this to equal 5, which means that we must have
\[1 + (6 - b)^2 = 25.\]
Subtracting 1 from both sides of the equation above gives
\[(6 - b)^2 = 24,\]
which implies that \(6 - b = \pm \sqrt{24}.\) Thus \(b = 6 - \sqrt{24}\) or \(b = 6 + \sqrt{24}.\)

17. Find two points on the horizontal axis whose distance from \((3, 2)\) equals 7.

**SOLUTION** A typical point on the horizontal axis has coordinates \((x, 0).\) The distance from this point to \((3, 2)\) is \(\sqrt{(x - 3)^2 + (0 - 2)^2}.\) Thus we need to solve the equation
\[\sqrt{(x - 3)^2 + 4} = 7.\]
Squaring both sides of the equation above, and then subtracting 4 from both sides gives
\[(x - 3)^2 = 45.\]
Thus \(x - 3 = \pm \sqrt{45} = \pm 3\sqrt{5}.\) Thus \(x = 3 \pm 3\sqrt{5}.\) Hence the two points on the horizontal axis whose distance from \((3, 2)\) equals 7 are \((3 + 3\sqrt{5}, 0)\) and \((3 - 3\sqrt{5}, 0).\)

19. A ship sails north for 2 miles and then west for 5 miles. How far is the ship from its starting point?

**SOLUTION** The figure below shows the path of the ship. The length of the red line is the distance of the ship from its starting point. By the Pythagorean Theorem, this distance is \(\sqrt{2^2 + 5^2}\) miles, which equals \(\sqrt{29}\) miles.

We have assumed that the surface of the Earth is part of a plane rather than part of a sphere. For distances of less than a few hundred miles, this is a good approximation.

**Use the following information concerning Manhattan (New York City) for Exercises 21–22:**

- **Avenues in Manhattan run roughly north-south; streets run east-west.**
- **For much of its length, Broadway runs diagonally across the grid formed by avenues and streets.**
- **The distance between consecutive avenues in Manhattan is 922 feet.**
- **The distance between consecutive streets in Manhattan is 260 feet.**

21. Suppose you walk from the corner of Central Park at 8th Avenue and 59th Street to 10th Avenue and 71st Street.

(a) What is the length of your path if you walk along 8th Avenue to 71st Street, then along 71st Street to 10th Avenue?
(b) What is the length of your path if you walk along Broadway, which goes in a straight line from 8th Avenue and 59th Street to 10th Avenue and 71st Street?
(c) How much shorter is the direct path along Broadway than walking along 8th Avenue and 71st Street?
(d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 8th Avenue and 71st Street?
(a) Walking along 8th Avenue from 59th Street to 71st Street is 12 blocks (because 71 - 59 = 12), each of which is 260 feet, for a total of 12 \times 260 feet, which equals 3120 feet. Walking along 71st Street from 8th Avenue to 10th Avenue is 2 blocks, each of which is 922 feet, for a total of 1844 feet. Thus the total path consists of 3120 feet plus 1844 feet, which equals 4964 feet.

(b) The length of the path along Broadway is the length of the hypotenuse of a right triangle whose other sides have length 3120 feet and 1844 feet, as calculated in part (a). Thus the length of the path along Broadway is

\[ \sqrt{3120^2 + 1844^2} = \sqrt{13134736} \approx 3624 \text{ feet}. \]

(c) Subtracting the results of part (b) from the result of part (a), we see that the path along Broadway is about 1340 feet shorter. Because a mile contains 5280 feet, the path along Broadway is about a quarter-mile shorter

\( \frac{1340}{5280} \approx 0.25 \).

(d) Part (c) shows that the path along Broadway is about 1340 feet shorter. Thus the amount of time saved is 1340 \times \frac{1}{2} = 670 \text{ minutes}, which equals a bit more than 5 minutes.

23. Find two points on the vertical axis whose distance from (5, −1) equals 8.

**SOLUTION** A typical point on the vertical axis has coordinates (0, y). The distance from this point to (5, −1) is \( \sqrt{(0-5)^2 + (y - (-1))^2} \). Thus we need to solve the equation

\[ 25 + (y + 1)^2 = 8. \]

Squaring both sides of the equation above, and then subtracting 25 from both sides gives

\[ (y + 1)^2 = 39. \]

Thus \( y + 1 = \pm \sqrt{39} \). Thus \( y = -1 \pm \sqrt{39} \). Hence the two points on the vertical axis whose distance from (5, −1) equals 8 are \( (0, -1 + \sqrt{39}) \) and \( (0, -1 - \sqrt{39}) \).

25. Find the perimeter of the triangle that has vertices at \((1,2), (5, -3), \) and \((-4, -1)\).

**SOLUTION** The perimeter of the triangle equals the sum of the lengths of the three sides of the triangle. Thus we find the lengths of those three sides.

The side of the triangle connecting the vertices (1, 2) and (5, −3) has length

\[ \sqrt{(5-1)^2 + (-3-2)^2} = \sqrt{41}. \]

The side of the triangle connecting the vertices (5, −3) and (−4, −1) has length

\[ \sqrt{(-4-5)^2 + (-1-(-3))^2} = \sqrt{85}. \]

The side of the triangle connecting the vertices (−4, −1) and (1, 2) has length

\[ \sqrt{(1-(-4))^2 + (2-(-1))^2} = \sqrt{34}. \]

Thus the perimeter of the triangle equals \( \sqrt{41} + \sqrt{85} + \sqrt{34} \).

27. Find the radius of a circle that has circumference 12 inches.

**SOLUTION** Let \( r \) denote the radius of this circle in inches. Thus \( 2\pi r = 12 \), which implies that \( r = \frac{6}{\pi} \) inches.

29. Find the radius of a circle that has circumference 8 more than its diameter.

**SOLUTION** Let \( r \) denote the radius of this circle. Thus the circle has circumference \( 2\pi r \) and has diameter \( 2r \). Because the circumference is 8 more than diameter, we have \( 2\pi r = 2r + 8 \). Thus \( 2(\pi - 2)r = 8 \), which implies that \( r = \frac{4}{\pi - 2} \).

31. Find an equation whose graph in the \( xy \)-plane is the set of points whose distance from the origin is 3.

**SOLUTION** The distance from a point \((x, y)\) to the origin is \( \sqrt{x^2 + y^2} \). Thus the equation we seek is

\[ \sqrt{x^2 + y^2} = 3. \]

To write this equation without using square roots, square both sides to get the equivalent equation

\[ x^2 + y^2 = 9. \]
33. Find the length of the graph of the equation \( x^2 + y^2 = 9 \).

**SOLUTION** From Exercise 31, we see that the graph of \( x^2 + y^2 = 9 \) is the set of points in the \( xy \)-plane whose distance from the origin is 3. In other words, the graph of \( x^2 + y^2 = 9 \) is the circle of radius 3 centered at the origin. The length (or circumference) of this graph is \( 2\pi \cdot 3 \), which equals \( 6\pi \).

35. Find an equation whose graph in the \( rt \)-plane is the set of points whose distance from \((3, 2)\) is 2.

**SOLUTION** The distance from a point \((r, t)\) to \((3, 2)\) is \( \sqrt{(r - 3)^2 + (t - 2)^2} \). Thus the equation we seek is
\[
\sqrt{(r - 3)^2 + (t - 2)^2} = 2.
\]

To write this equation without using square roots, square both sides to get the equivalent equation
\[
(r - 3)^2 + (t - 2)^2 = 4.
\]

37. Find two points that have distance 2 from the origin and distance 3 from \((3, 0)\).

**SOLUTION** Suppose \((x, y)\) has distance 2 from the origin and distance 3 from \((3, 0)\). Thus
\[
\sqrt{x^2 + y^2} = 2 \quad \text{and} \quad \sqrt{(x - 3)^2 + y^2} = 3.
\]

Squaring both sides of these equations gives
\[
x^2 + y^2 = 4 \quad \text{and} \quad x^2 - 6x + 9 + y^2 = 9.
\]

Subtracting the second equation from the first equation gives
\[
6x = 4.
\]

Thus \( x = \frac{2}{3} \). Substituting this value of \( x \) into the equation \( x^2 + y^2 = 4 \) gives the equation \( y^2 = \frac{4}{3} \). Thus \( y = \pm \frac{\sqrt{4}}{\sqrt{3}} = \pm \frac{2}{\sqrt{3}} \). Hence the two points we seek are \((\frac{2}{3}, \frac{2}{\sqrt{3}})\) and \((\frac{2}{3}, -\frac{2}{\sqrt{3}})\).

38. Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal half the total length of the straight part of the track. What dimensions should the track have?

**SOLUTION** Let \( t \) equal the total length of the straight part of the track in meters. We want the curved part of the track to have total length \( \frac{t}{2} \). Thus we want
\[
t + \frac{t}{2} = 400.
\]

Solving this equation for \( t \), we get \( t = \frac{800}{3} \) meters. Thus each of the two straight pieces must be \( \frac{400}{3} \) meters long. Hence we take \( c = \frac{400}{3} \approx 133.33 \) meters in the figure below.

![Diagram](d-c)

We also want the total length of the two half-circles to be \( \frac{400}{3} \) meters. Thus we want \( \pi d = \frac{400}{3} \). Hence we take \( d = \frac{400}{3\pi} \approx 42.44 \) meters in the figure above.

41. Suppose a rope is just long enough to cover the equator of the Earth. About how much longer would the rope need to be so that it could be suspended seven feet above the entire equator?

**SOLUTION** Assume that the equator of the Earth is a circle. This assumption is close enough to being correct to answer a question that requires only an approximation.

Assume that the radius of the Earth is \( r \), measured in feet (note that we do not need to know the value of \( r \) for this exercise). For a rope to cover the equator, it needs to have length \( 2\pi r \) feet. For a rope to be suspended seven feet above the equator, it would need to have length \( 2\pi (r + 7) \) feet, which equals \( (2\pi r + 14\pi) \) feet. In other words, to be suspended seven feet above the equator, the rope would need to be only \( 14\pi \) feet longer than a rope covering the equator. Because \( 14\pi \approx 44 \cdot \frac{22}{7} = 44 \), the rope would need to be about 44 feet longer than a rope covering the equator.


2.2 Lines

**LEARNING OBJECTIVES**

By the end of this section you should be able to
- find the slope of a line;
- find the equation of a line given its slope and a point on it;
- find the equation of a line given two points on it;
- determine whether or not two lines are parallel;
- find the equation of a line perpendicular to a given line and containing a given point;
- find the midpoint of a line segment.

**Slope**

Consider a line in the coordinate plane, along with four points \((x_1, y_1)\), \((x_2, y_2)\), \((x_3, y_3)\), and \((x_4, y_4)\) on the line. Draw two right triangles with horizontal and vertical edges as in the figure below:

![Similar triangles.](image)

The two right triangles in the figure above are similar because their angles are equal. Thus the ratios of the corresponding sides of the two triangles above are equal. Specifically, taking the ratio of the vertical side and horizontal side for each triangle, we have

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}.
\]

The equation above states that for any pair of points \((x_1, y_1)\) and \((x_2, y_2)\) on the line, the ratio \(\frac{y_2 - y_1}{x_2 - x_1}\) does not depend on the particular pair of points chosen on the line. If we choose another pair of points on the line, say \((x_3, y_3)\) and \((x_4, y_4)\) instead of \((x_1, y_1)\) and \((x_2, y_2)\), then the difference of second coordinates divided by the difference of first coordinates remains the same, as shown by the equation above.

Thus the ratio \(\frac{y_2 - y_1}{x_2 - x_1}\) is a constant depending only on the line and not on the particular points \((x_1, y_1)\) and \((x_2, y_2)\) chosen on the line. This constant is called the **slope** of the line.
If \((x_1, y_1)\) and \((x_2, y_2)\) are any two points on a line, with \(x_1 \neq x_2\), then the **slope** of the line is

\[
\frac{y_2 - y_1}{x_2 - x_1}.
\]

**EXAMPLE 1**

Find the slope of the line containing the points \((2, 1)\) and \((5, 3)\).

**SOLUTION** The line containing \((2, 1)\) and \((5, 3)\) is shown here. The slope of this line is \(\frac{3-1}{5-2}\), which equals \(\frac{2}{3}\).

---

A line with positive slope slants up from left to right; a line with negative slope slants down from left to right. Lines whose slopes have larger absolute value are steeper than lines whose slopes have smaller absolute value. This figure shows some lines and their slopes; the same scale has been used on both axes.

In the figure above, the horizontal axis has slope 0, as does every horizontal line. Vertical lines, including the vertical axis, do not have a slope, because a vertical line does not contain two points \((x_1, y_1)\) and \((x_2, y_2)\) with \(x_1 \neq x_2\).

**The Equation of a Line**

Consider a line with slope \(m\), and suppose \((x_1, y_1)\) is a point on this line. Let \((x, y)\) denote a typical point on the line, as shown here.

Because this line has slope \(m\), we have

\[
\frac{y - y_1}{x - x_1} = m.
\]

Multiplying both sides of the equation above by \(x - x_1\), we get the following formula:
The equation of a line, given its slope and one point on it

The line in the \(xy\)-plane that has slope \(m\) and contains the point \((x_1, y_1)\) is given by the equation

\[
y - y_1 = m(x - x_1).
\]

The symbol \(m\) is often used to denote the slope of a line.

The equation above can be solved for \(y\) to get an equation for the line in the form \(y = mx + b\), where \(m\) and \(b\) are constants.

---

**Example 2**

Find the equation of the line in the \(xy\)-plane that has slope \(\frac{1}{2}\) and contains \((4, 1)\).

**Solution**

In this case the equation displayed above becomes

\[
y - 1 = \frac{1}{2}(x - 4).
\]

Adding 1 to both sides and simplifying, we get

\[
y = \frac{1}{2}x - 1.
\]

As a check for possible errors, if we take \(x = 4\) in the equation above, we get \(y = 1\). Thus the point \((4, 1)\) is indeed on this line.

---

As a special case of finding the equation of a line when given its slope and one point on it, suppose we want to find the equation of the line in the \(xy\)-plane with slope \(m\) that intersects the \(y\)-axis at the point \((0, b)\). In this case, the formula above becomes

\[
y - b = m(x - 0).
\]

Solving this equation for \(y\), we have the following result:

The equation of a line, given its slope and \(y\)-intercept

The line in the \(xy\)-plane with slope \(m\) that intersects the \(y\)-axis at \((0, b)\) is given by the equation

\[
y = mx + b.
\]

---

**Example 3**

Find the equation of the line in the \(xy\)-plane that has slope \(\frac{1}{2}\) and intersects the \(y\)-axis at \((0, -1)\).

**Solution**

The formula above shows that the desired equation is

\[
y = \frac{1}{2}x - 1.
\]

This line is shown in Example 2, where you can see that the line intersects the \(y\)-axis at \((0, -1)\).
If a line contains the origin, then \( b = 0 \) in the formula above. For example, the line in the \( xy \)-plane that has slope 2 and contains the origin is given by the equation \( y = 2x \). The figure below Example 1 shows the line \( y = 2x \) and several other lines containing the origin.

Suppose now that we want to find the equation of the line containing two specific points. We can reduce this problem to a problem we have already solved by computing the slope of the line and then using the formula in the box above.

Specifically, suppose we want to find the equation of the line containing the points \((x_1, y_1)\) and \((x_2, y_2)\), where \( x_1 \neq x_2 \). This line has slope \( (y_2 - y_1)/(x_2 - x_1) \). Thus our formula for the equation of a line when given its slope and one point on it gives the following result:

**The equation of a line, given two points on it**

The line in the \( xy \)-plane that contains the points \((x_1, y_1)\) and \((x_2, y_2)\), where \( x_1 \neq x_2 \), is given by the equation

\[
y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).
\]

**EXAMPLE 4** Find the equation of the line in the \( xy \)-plane that contains the points \((2, 4)\) and \((5, 1)\).

**SOLUTION** In this case the equation above becomes

\[
y - 4 = \frac{1 - 4}{5 - 2}(x - 2).
\]

Solving this equation for \( y \), we get

\[
y = -x + 6.
\]

As a check, if we take \( x = 2 \) in the equation above, we get \( y = 4 \), and if we take \( x = 5 \) in the equation above, we get \( y = 1 \); thus the points \((2, 4)\) and \((5, 1)\) are indeed on this line.

Conversion between different units of measurement is usually done by an equation that represents one unit as a suitable multiple of another unit. For example, a pound is officially defined to be exactly 0.45359237 kilograms. Thus the equation that gives the weight \( k \) in kilograms for an object weighing \( p \) pounds is

\[
k = 0.45359237p.
\]

Conversion between temperature scales is unusual because the zero temperature on one scale does not correspond to the zero temperature on
another scale. Most other quantities such weights, lengths, and currency have the same zero point regardless of the units used. For example, without knowing the conversion rate, you know that 0 centimeters is the same length as 0 inches.

The next example shows how to find a formula for converting from Celsius temperatures to Fahrenheit temperatures. The solution to the next example also shows that rather than memorizing the formula, for example, of the equation of a line given two points on it, sometimes it is simpler just to use the available information to find the constants $m$ and $b$ that characterize the line $y = mx + b$.

Find an equation that gives the temperature $F$ on the Fahrenheit scale corresponding to temperature $C$ on the Celsius scale.

**SOLUTION** We seek an equation of the form

$$F = mC + b$$

for some constants $m$ and $b$.

To find $m$ and $b$, we start by recalling that the freezing temperature of water equals 0 degrees Celsius and 32 degrees Fahrenheit. Plugging $C = 0$ and $F = 32$ into the equation above gives $32 = b$. Now that we know that $b = 32$, the equation above can be rewritten as

$$F = mC + 32.$$  

Recall now that the boiling point of water equals 100 degrees Celsius and 212 degrees Fahrenheit. Plugging $C = 100$ and $F = 212$ into the equation above and then solving for $m$ shows that $m = \frac{9}{5}$. Thus the formula we seek is

$$F = \frac{9}{5}C + 32.$$  

**Parallel Lines**

Consider two parallel lines in the coordinate plane, as shown in the figure below:

The graph of $F = \frac{9}{5}C + 32$ on the interval $[-10, 110]$.  

This thermometer shows Celsius degrees on the left, Fahrenheit degrees on the right.
Because the two lines are parallel, the corresponding angles in the two triangles above are equal (as shown by the arcs in the figure above), and thus the two right triangles are similar. This implies that

\[ \frac{b}{a} = \frac{d}{c}. \]

Because \( \frac{b}{a} \) is the slope of the top line and \( \frac{d}{c} \) is the slope of the bottom line, we conclude that these parallel lines have the same slope.

The logic used in the paragraph above is reversible. Specifically, suppose instead of starting with the assumption that the two lines in the figure above are parallel, we start with the assumption that the two lines have the same slope. Thus \( \frac{b}{a} = \frac{d}{c} \), which implies that the two right triangles in the figure above are similar. Hence the two lines make equal angles with the horizontal axis, as shown by the arcs in the figure, which implies that the two lines are parallel.

The figure and reasoning given above do not work if both lines are horizontal or both lines are vertical. But horizontal lines all have slope 0, and the slope is not defined for vertical lines. Thus we can summarize our characterization of parallel lines as follows:

**Parallel lines**

Two nonvertical lines in the coordinate plane are parallel if and only if they have the same slope.

---

**EXAMPLE 6**

(a) Are the lines in the \( xy \)-plane given by the following equations parallel?

\[ y = 4x - 5 \quad \text{and} \quad y = 4x + 18 \]

(b) Are the lines in the \( xy \)-plane given by the following equations parallel?

\[ y = 6x + 5 \quad \text{and} \quad y = 7x + 5 \]

**SOLUTION**

(a) These lines are parallel because they have the same slope (which equals 4).

(b) These lines are not parallel because their slopes are not equal—the first line has slope 6 and the second line has slope 7.

---

**Perpendicular Lines**

Before beginning our treatment of perpendicular lines, we take a brief detour to make clear the geometry of a line with negative slope. A line with negative slope slants down from left to right. The figure below shows a line with negative slope; to avoid clutter the coordinate axes are not shown.
In the figure above, \( a \) is the length of the horizontal line segment and \( c \) is the length of the vertical line segment. Of course \( a \) and \( c \) are positive numbers, because lengths are positive. In terms of the coordinates as shown in the figure above, we have \( a = x_2 - x_1 \) and \( c = y_1 - y_2 \). The slope of this line equals \((y_2 - y_1)/(x_2 - x_1)\), which equals \(-c/a\).

The following result gives a useful characterization of perpendicular lines:

**Perpendicular lines**

Two nonvertical lines are perpendicular if and only if the product of their slopes equals \(-1\).

To explain why the result above holds, consider two perpendicular lines as shown in blue in the figure here. In addition to the two perpendicular lines in blue, the figure shows the horizontal line segment \(PS\) and the vertical line segment \(QT\), which intersect at \(S\).

We assume that the angle \(PQT\) is \(\theta\) degrees. To check that the other three labeled angles in the figure are labeled correctly, first note that the two angles labeled \(90 - \theta\) are each the third angle in a right triangle (the right triangles are \(PSQ\) and \(QPT\)), one of whose angles is \(\theta\). Consideration of the right angle \(QPT\) now shows that angle \(TPS\) is \(\theta\) degrees, as labeled.

The line containing the points \(P\) and \(Q\) has slope \(b/a\), as can be seen from the figure. Furthermore, our brief discussion of lines with negative slope shows that the line containing the points \(P\) and \(T\) has slope \(-c/a\).

Consider the right triangles \(PSQ\) and \(TSP\) in the figure. These triangles have the same angles, and thus they are similar. Thus the ratios of corresponding sides are equal. Specifically, we have

\[
\frac{b}{a} = \frac{a}{c}.
\]

Multiplying both sides of this equation by \(-c/a\), we get

\[
\left(\frac{b}{a}\right) \cdot \left(-\frac{c}{a}\right) = -1.
\]

As we have already seen, the first quantity on the left above is the slope of the line containing the points \(P\) and \(Q\), and the second quantity is the slope of the line containing the points \(P\) and \(T\). Thus we can conclude that the product of the slopes of these two perpendicular lines equals \(-1\), as desired.

The logic used above is reversible. Specifically, suppose instead of starting with the assumption that the two lines in blue are perpendicular, we start
with the assumption that the product of their slopes equals $-1$. This implies that $\frac{b}{a} = \frac{\alpha}{c}$, which implies that the two right triangles $PSQ$ and $TSP$ are similar; thus these two triangles have the same angles. This implies that the angles are labeled correctly in the figure above (assuming that we start by declaring that angle $PQS$ measures $\theta$ degrees). This then implies that angle $QPT$ measures $90^\circ$. Thus the two lines in blue are perpendicular, as desired.

**EXAMPLE 7**

Show the lines in the $xy$-plane given by the following equations are perpendicular:

$$y = 4x - 5 \quad \text{and} \quad y = -\frac{1}{4}x + 18$$

**SOLUTION**

The first line has slope 4; the second line has slope $-\frac{1}{4}$. The product of these slopes is $4 \cdot (-\frac{1}{4})$, which equals $-1$. Because the product of the two slopes equals $-1$, the two lines are perpendicular.

To show that two lines are perpendicular, we only need to know the slopes of the lines, not their full equations, as shown by the following example.

**EXAMPLE 8**

Show that the line containing the points $(1, -2)$ and $(3, 3)$ is perpendicular to the line containing the points $(9, -1)$ and $(4, 1)$.

**SOLUTION**

The line containing $(1, -2)$ and $(3, 3)$ has slope $\frac{3 - (-2)}{3 - 1}$, which equals $\frac{5}{3}$. Also, the line containing $(9, -1)$ and $(4, 1)$ has slope $\frac{1 - (-1)}{4 - 9}$, which equals $\frac{-2}{5}$. Because the product $\frac{5}{3} \cdot (-\frac{2}{5})$ equals $-1$, the two lines are perpendicular.

**Midpoints**

This subsection begins with an intuitive definition of the midpoint of a line segment:

**Midpoint**

The midpoint of a line segment is the point on the line segment that lies halfway between the two endpoints.
As you might guess, the first coordinate of the midpoint of a line segment is the average of the first coordinates of the endpoints. Similarly, the second coordinate of the midpoint is the average of the second coordinates of the endpoints. Here is the formal statement of this formula:

**Midpoint**

The midpoint of the line segment connecting \((x_1, y_1)\) and \((x_2, y_2)\) equals \(\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)\).

The next example illustrates the use of the formula above.

(a) Find the midpoint of the line segment connecting \((1, 3)\) and \((5, 9)\).

(b) Verify that the distance between the midpoint found in (a) and the first endpoint \((1, 3)\) equals the distance between the midpoint found in (a) and the second endpoint \((5, 9)\).

(c) Verify that the midpoint found in (a) lies on the line connecting \((1, 3)\) and \((5, 9)\).

**SOLUTION**

(a) Using the formula above, we see that the midpoint of the line segment connecting \((1, 3)\) and \((5, 9)\) equals

\(\left( \frac{1 + 5}{2}, \frac{3 + 9}{2} \right)\),

which equals \((3, 6)\).

(b) First compute the distance between the midpoint and the endpoint \((1, 3)\):

\[\text{distance between } (3, 6) \text{ and } (1, 3) = \sqrt{(3 - 1)^2 + (6 - 3)^2} = \sqrt{2^2 + 3^2} = \sqrt{13}.\]

Next compute the distance between the midpoint and the endpoint \((5, 9)\):

\[\text{distance between } (3, 6) \text{ and } (5, 9) = \sqrt{(3 - 5)^2 + (6 - 9)^2} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}.\]

As expected, these two distances are equal—the distance between the midpoint and either endpoint is \(\sqrt{13}\).

(c) First compute the slope of the line containing the midpoint and the endpoint \((1, 3)\):

\[\text{slope of line containing } (3, 6) \text{ and } (1, 3) = \frac{6 - 3}{3 - 1} = \frac{3}{2}.\]

Next compute the slope of the line containing the midpoint and the endpoint \((5, 9)\):

Problems 51–53 at the end of this section will lead you to an explanation of why this formula for the midpoint is correct.
As expected, these two slopes are equal. In other words, the line segment from 
(1, 3) to (3, 6) points in the same direction as the line segment from 
(3, 6) to (5, 9). Thus these three points all lie on the same line.

**EXERCISES**

1. What are the coordinates of the unlabeled vertex of the smaller of the two right triangles in the figure at the beginning of this section?

2. What are the coordinates of the unlabeled vertex of the larger of the two right triangles in the figure at the beginning of this section?

3. Find the slope of the line that contains the points (3, 4) and (7, 13).

4. Find the slope of the line that contains the points (2, 11) and (6, −5).

5. Find a number \( t \) such that the line containing the points \((1, t)\) and \((3, 7)\) has slope 5.

6. Find a number \( c \) such that the line containing the points \((c, 4)\) and \((-2, 9)\) has slope −3.

7. Suppose the tuition per semester at Euphoria State University is $525 plus $200 for each class unit taken.

   (a) Find an equation that gives the tuition \( t \) in dollars for taking \( u \) class units.

   (b) Find the total tuition for accumulating 120 units over 8 semesters.

   (c) Find the total tuition for accumulating 120 units over 12 semesters.

8. Suppose the tuition per semester at Luxim University is $900 plus $850 for each class unit taken.

   (a) Find an equation that gives the tuition \( t \) in dollars for taking \( u \) class units.

   (b) Find the total tuition for accumulating 120 units over 8 semesters.

   (c) Find the total tuition for accumulating 120 units over 10 semesters.

9. Suppose your cell phone company offers two calling plans. The pay-per-call plan charges $14 per month plus 3 cents for each minute. The unlimited-calling plan charges a flat rate of $29 per month for unlimited calls.

   (a) Find an equation that gives the cost \( c \) in dollars for making \( m \) minutes of phone calls per month on the pay-per-call plan.

   (b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

10. Suppose your cell phone company offers two calling plans. The pay-per-call plan charges $11 per month plus 4 cents for each minute. The unlimited-calling plan charges a flat rate of $25 per month for unlimited calls.

    (a) Find an equation that gives the cost \( c \) in dollars for making \( m \) minutes of phone calls per month on the pay-per-call plan.

    (b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

11. Find the equation of the line in the \( xy \)-plane with slope 2 that contains the point \((7, 3)\).

12. Find the equation of the line in the \( xy \)-plane with slope −4 that contains the point \((-5, -2)\).

13. Find the equation of the line that contains the points \((2, -1)\) and \((4, 9)\).

14. Find the equation of the line that contains the points \((-3, 2)\) and \((-5, 7)\).

15. Find a number \( t \) such that the point \((3, t)\) is on the line containing the points \((7, 6)\) and \((14, 10)\).

16. Find a number \( t \) such that the point \((-2, t)\) is on the line containing the points \((5, -2)\) and \((10, -8)\).
17. Find a formula for the number of seconds \( s \) in \( d \) days.
18. Find a formula for the number of seconds \( s \) in \( w \) weeks.
19. Find a formula for the number of inches \( I \) in \( M \) miles.
20. Find a formula for the number of miles \( M \) in \( F \) feet.
21. Find a formula for the number of kilometers \( k \) in \( M \) miles.

\[ \text{The exact conversion between the English measurement system and the metric system is given by the equation } 1 \text{ inch} = 2.54 \text{ centimeters}. \]
22. Find a formula for the number of miles \( M \) in \( m \) meters.
23. Find a formula for the number of inches \( I \) in \( c \) centimeters.
24. Find a formula for the number of inches \( I \) in \( c \) centimeters.
25. Find a number \( c \) such that the point \((c, 13)\) is on the line containing the points \((-4, -17)\) and \((6, 33)\).
26. Find a number \( c \) such that the point \((c, -19)\) is on the line containing the points \((2, 1)\) and \((4, 9)\).
27. Find a number \( t \) such that the point \((t, 2t)\) is on the line containing the points \((3, -7)\) and \((5, -15)\).
28. Find a number \( t \) such that the point \((t, \frac{t}{2})\) is on the line containing the points \((2, -4)\) and \((-3, -11)\).
29. Find the equation of the line in the \( xy \)-plane that contains the point \((3, 2)\) and that is parallel to the line \( y = 4x - 1 \).
30. Find the equation of the line in the \( xy \)-plane that contains the point \((-4, -5)\) and that is parallel to the line \( y = -2x + 3 \).
31. Find the equation of the line that contains the point \((2, 3)\) and that is parallel to the line containing the points \((7, 1)\) and \((5, 6)\).
32. Find the equation of the line that contains the point \((-4, 3)\) and that is parallel to the line containing the points \((3, -7)\) and \((6, -9)\).
33. Find a number \( t \) such that the line containing the points \((t, 2)\) and \((3, 5)\) is parallel to the line containing the points \((-1, 4)\) and \((-3, -2)\).
34. Find a number \( t \) such that the line containing the points \((-3, t)\) and \((2, -4)\) is parallel to the line containing the points \((5, 6)\) and \((-2, 4)\).
35. Find the intersection in the \( xy \)-plane of the lines \( y = 5x + 3 \) and \( y = -2x + 1 \).
36. Find the intersection in the \( xy \)-plane of the lines \( y = -4x + 5 \) and \( y = 5x - 2 \).
37. Find a number \( b \) such that the three lines in the \( xy \)-plane given by the equations \( y = 2x + b \), \( y = 3x - 5 \), and \( y = -4x + 6 \) have a common intersection point.
38. Find a number \( m \) such that the three lines in the \( xy \)-plane given by the equations \( y = mx + 3 \), \( y = 4x + 1 \), and \( y = 5x + 7 \) have a common intersection point.
39. Find the equation of the line in the \( xy \)-plane that contains the point \((4, 1)\) and that is perpendicular to the line whose equation is \( y = 3x + 5 \).
40. Find the equation of the line in the \( xy \)-plane that contains the point \((-3, 2)\) and that is perpendicular to the line whose equation is \( y = -5x + 1 \).
41. Find a number \( t \) such that the line in the \( xy \)-plane containing the points \((t, 4)\) and \((2, -1)\) is perpendicular to the line \( y = 6x - 7 \).
42. Find a number \( t \) such that the line in the \( xy \)-plane containing the points \((-3, t)\) and \((4, 3)\) is perpendicular to the line \( y = -5x + 999 \).
43. Find a number \( t \) such that the line containing the points \((4, t)\) and \((-1, 6)\) is perpendicular to the line that contains the points \((3, 5)\) and \((1, -2)\).
44. Find a number \( t \) such that the line containing the points \((t, -2)\) and \((-3, 5)\) is perpendicular to the line that contains the points \((4, 7)\) and \((1, 11)\).
45. Find the midpoint of the line segment connecting \((-3, 4)\) and \((5, 7)\).
46. Find the midpoint of the line segment connecting \((6, -5)\) and \((-3, -8)\).
47. Find numbers \( x \) and \( y \) such that \((-2, 5)\) is the midpoint of the line segment connecting \((3, 1)\) and \((x, y)\).
48. Find numbers \( x \) and \( y \) such that \((3, -4)\) is the midpoint of the line segment connecting \((-2, 5)\) and \((x, y)\).
PROBLEMS

49. The Kelvin temperature scale is defined by
   \( K = C + 273.15 \), where \( K \) is the temperature
   on the Kelvin scale and \( C \) is the temperature
   on the Celsius scale. (Thus \(-273.15 \) degrees
   Celsius, which is the temperature at which all
   molecular movement ceases and thus is the
   lowest possible temperature, corresponds to 0
   on the Kelvin scale.)
   
   (a) Find an equation that gives the tempera-
       ture \( F \) on the Fahrenheit scale corre-
       sponding to temperature \( K \) on the Kelvin scale.
   
   (b) Explain why the graph of the equation
       from part (a) is parallel to the graph of the
       equation obtained in Example 5.

50. Find the equation of the line in the \( xy \)-plane
    that has slope \( m \) and intersects the \( x \)-axis at
    \((c, 0)\).

51. Suppose \((x_1, y_1)\) and \((x_2, y_2)\) are the endpoints
    of a line segment.
    
    (a) Show that the distance between the point
        \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\) and the endpoint
        \((x_1, y_1)\) equals half the length of the line segment.
    
    (b) Show that the distance between the point
        \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\) and the endpoint
        \((x_2, y_2)\) equals half the length of the line segment.

52. Suppose \((x_1, y_1)\) and \((x_2, y_2)\) are the endpoints
    of a line segment.
    
    (a) Show that the line containing the point
        \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\) and the endpoint
        \((x_1, y_1)\) has slope \(\frac{y_2-y_1}{x_2-x_1}\).
    
    (b) Show that the line containing the point
        \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\) and the endpoint
        \((x_2, y_2)\) has slope \(\frac{y_2-y_1}{x_2-x_1}\).
    
    (c) Explain why parts (a) and (b) of this problem imply
        that the point \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\)
        lies on the line containing the endpoints
        \((x_1, y_1)\) and \((x_2, y_2)\).

53. Explain why the two previous problems imply
    that \((\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})\) is the midpoint of the line
    segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\).

54. Find six distinct points on the circle with center
    \((2, 3)\) and radius 5.

55. Find six distinct points on the circle with center
    \((4, 1)\) and circumference 3.

56. We used similar triangles to show that the prod-
    uct of the slopes of two perpendicular lines
    equals \(-1\). The steps below outline an alter-
    native proof that avoids the use of similar tri-
    angles but uses more algebra instead. Use the
    figure below, which is the same as the figure
    used earlier except that there is now no need to
    label the angles.

\[ QP \text{ is perpendicular to } PT. \]

(a) Apply the Pythagorean Theorem to tri-
    angle \(PSQ\) to find the length of the line
    segment \(PQ\) in terms of \(a\) and \(b\).

(b) Apply the Pythagorean Theorem to tri-
    angle \(PST\) to find the length of the line
    segment \(PT\) in terms of \(a\) and \(c\).

(c) Apply the Pythagorean Theorem to tri-
    angle \(QPT\) to find the length of the line
    segment \(QT\) in terms of the lengths of the
    line segments of \(PQ\) and \(PT\) calculated in
    the first two parts of this problem.

(d) As can be seen from the figure, the length
    of the line segment \(QT\) equals \(b + c\). Thus
    set the formula for length of the line segment
    \(QT\), as calculated in the previous
    part of this problem, equal to \(b + c\), and
    solve the resulting equation for \(c\) in terms
    of \(a\) and \(b\).

(e) Use the result in the previous part of this
    problem to show that the slope of the line
    containing \(P\) and \(Q\) times the slope of the
    line containing \(P\) and \(T\) equals \(-1\).
57. Suppose $a$ and $b$ are nonzero numbers. Where does the line in the $xy$-plane given by the equation \[
\frac{x}{a} + \frac{y}{b} = 1
\] intersect the coordinate axes?

58. Show that the points $(-84, -14)$, $(21, 1)$, and $(98, 12)$ lie on a line.

59. Show that the points $(-8, -65)$, $(1, 52)$, and $(3, 77)$ do not lie on a line.

60. Change just one of the six numbers in the problem above so that the resulting three points do lie on a line.

61. Show that for every number $t$, the point \[
(5 - 3t, 7 - 4t)
\] is on the line containing the points $(2, 3)$ and $(5, 7)$.

WORKED-OUT SOLUTIONS to Odd-numbered Exercises

1. What are the coordinates of the unlabeled vertex of the smaller of the two right triangles in the figure at the beginning of this section?

**SOLUTION** Drawing vertical and horizontal lines from the point in question to the coordinate axes shows that the coordinates of the point are $(x_2, y_1)$.

3. Find the slope of the line that contains the points $(3, 4)$ and $(7, 13)$.

**SOLUTION** The line containing the points $(3, 4)$ and $(7, 13)$ has slope \[
\frac{13 - 4}{7 - 3} = \frac{9}{4},
\] which equals $\frac{9}{4}$.

5. Find a number $t$ such that the line containing the points $(1, t)$ and $(3, 7)$ has slope $5$.

**SOLUTION** The slope of the line containing the points $(1, t)$ and $(3, 7)$ equals \[
\frac{7 - t}{3 - 1} = \frac{7 - t}{2},
\] which equals $\frac{7 - t}{2}$. We want this slope to equal $5$. Thus we must find a number $t$ such that \[
\frac{7 - t}{2} = 5.
\] Solving this equation for $t$, we get $t = -3$.

7. Suppose the tuition per semester at Euphoria State University is $525 plus $200 for each class unit taken.

(a) Find an equation that gives the cost $c$ in dollars for making $m$ minutes of phone calls per month on the pay-per-call plan.

(b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

**SOLUTION**

(a) As usual, units must be consistent throughout any calculation. Thus we think of 3 cents as 0.03 dollars. Hence the cost $c$ in dollars for making $m$ minutes of phone calls per month is given by the equation \[
c = 0.03m + 14.
\]
(b) Setting \( c = 29 \) in the equation above gives the equation \( 29 = 0.03m + 14. \) Thus
\[
m = \frac{29 - 14}{0.03} = \frac{15}{0.03} = \frac{1500}{3} = 500.
\]
Thus the two plans cost an equal amount if 500 minutes per month are used. If more than 500 minutes per month are used, then the unlimited-calling plan is cheaper.

11. Find the equation of the line in the \( xy \)-plane with slope 2 that contains the point \((7, 3)\).

**SOLUTION** If \((x, y)\) denotes a typical point on the line with slope 2 that contains the point \((7, 3)\), then
\[
\frac{y - 3}{x - 7} = 2.
\]
Multiplying both sides of this equation by \(x - 7\) and then adding 3 to both sides gives the equation
\[
y = 2x - 11.
\]

**CHECK** The line whose equation is \(y = 2x - 11\) has slope 2. We should also check that the point \((7, 3)\) is on this line. In other words, we need to verify the alleged equation
\[
3 = 2 \cdot 7 - 11.
\]
Simple arithmetic shows that this is indeed true.

13. Find the equation of the line that contains the points \((2, -1)\) and \((4, 9)\).

**SOLUTION** The line that contains the points \((2, -1)\) and \((4, 9)\) has slope
\[
\frac{9 - (-1)}{4 - 2} = \frac{10}{2} = 5.
\]
which equals 5. Thus if \((x, y)\) denotes a typical point on this line, then
\[
\frac{y - 9}{x - 4} = 5.
\]
Multiplying both sides of this equation by \(x - 4\) and then adding 9 to both sides gives the equation
\[
y = 5x - 11.
\]

**CHECK** We need to check that both \((2, -1)\) and \((4, 9)\) are on the line whose equation is \(y = 5x - 11\). In other words, we need to verify the alleged equations
\[
-1 = 5 \cdot 2 - 11 \quad \text{and} \quad 9 = 5 \cdot 4 - 11.
\]
Simple arithmetic shows that both alleged equations are indeed true.

15. Find a number \(t\) such that the point \((3, t)\) is on the line containing the points \((7, 6)\) and \((14, 10)\).

**SOLUTION** First we find the equation of the line containing the points \((7, 6)\) and \((14, 10)\). To do this, note that the line containing those two points has slope
\[
\frac{10 - 6}{14 - 7} = \frac{4}{7},
\]
which equals \(\frac{4}{7}\). Thus if \((x, y)\) denotes a typical point on this line, then
\[
\frac{y - 6}{x - 7} = \frac{4}{7}.
\]
Multiplying both sides of this equation by \(x - 7\) and then adding 6 gives the equation
\[
y = \frac{4}{7}x + 2.
\]
Now we can find a number \(t\) such that the point \((3, t)\) is on the line given by the equation above. To do this, in the equation above replace \(x\) by 3 and \(y\) by \(t\), getting
\[
t = \frac{4}{7} \cdot 3 + 2.
\]
Performing the arithmetic to compute the right side, we get \(t = \frac{26}{7}\).

**CHECK** We should check that all three points \((7, 6)\), \((14, 10)\), and \((3, \frac{26}{7})\) are on the line \(y = \frac{4}{7}x + 2\). In other words, we need to verify the alleged equations
\[
6 = \frac{4}{7} \cdot 7 + 2, \quad 10 = \frac{4}{7} \cdot 14 + 2, \quad \frac{26}{7} = \frac{4}{7} \cdot 3 + 2.
\]
Simple arithmetic shows that all three alleged equations are indeed true.

17. Find a formula for the number of seconds \(s\) in \(d\) days.

**SOLUTION** Each minute has 60 seconds, and each hour has 60 minutes. Thus each hour has
60 \times 60 \text{ seconds}, \text{ or } 3600 \text{ seconds}. \text{ Each day has } 24 \text{ hours; thus each day has } 24 \times 3600 \text{ seconds}, \text{ or } 86400 \text{ seconds. Thus}
\[ s = 86400d. \]

19. Find a formula for the number of inches \( I \) in \( M \) miles.

**SOLUTION** Each foot has 12 inches, and each mile has 5280 feet. Thus each mile has \( 5280 \times 12 \text{ inches}, \text{ or } 63360 \text{ inches. Thus} \]
\[ I = 63360M. \]

21. Find a formula for the number of kilometers \( k \) in \( M \) miles.

**SOLUTION** Multiplying both sides of the equation
\[ 1 \text{ inch} = 2.54 \text{ centimeters} \]
by 12 gives
\[ 1 \text{ foot} = 12 \times 2.54 \text{ centimeters} = 30.48 \text{ centimeters}. \]
Multiplying both sides of the equation above by 5280 gives
\[ 1 \text{ mile} = 5280 \times 30.48 \text{ centimeters} = 160934.4 \text{ centimeters} = 1609.344 \text{ meters} = 1.609344 \text{ kilometers}. \]
Multiplying both sides of the equation above by a number \( M \) shows that \( M \) miles = \( 1.609344M \) kilometers. In other words,
\[ k = 1.609344M. \]

*The formula above is exact. However, the approximation \( k = 1.61M \) is often used.*

23. Find a formula for the number of inches \( I \) in \( c \) centimeters.

**SOLUTION** Dividing both sides of the equation
\[ 1 \text{ inch} = 2.54 \text{ centimeters} \]
by 2.54 gives
\[ 1 \text{ centimeter} = \frac{1}{2.54} \text{ inches}. \]
Multiplying both sides of the equation above by a number \( c \) shows that \( c \) centimeters = \( \frac{c}{2.54} \) inches. In other words,
\[ I = \frac{c}{2.54}. \]

25. Find a number \( c \) such that the point \((c,13)\) is on the line containing the points \((-4,-17)\) and \((6,33)\).

**SOLUTION** First we find the equation of the line containing the points \((-4,-17)\) and \((6,33)\). To do this, note that the line containing those two points has slope
\[ \frac{33 - (-17)}{6 - (-4)}, \]
which equals 5. Thus if \((x, y)\) denotes a typical point on this line, then
\[ \frac{y - 33}{x - 6} = 5. \]
Multiplying both sides of this equation by \(x - 6\) and then adding 33 gives the equation
\[ y = 5x + 3. \]
Now we can find a number \( c \) such that the point \((c,13)\) is on the line given by the equation above. To do this, in the equation above replace \(x\) by \(c\) and \(y\) by 13, getting
\[ 13 = 5c + 3. \]
Solving this equation for \(c\), we get \(c = 2\).

**CHECK** We should check that the three points \((-4,-17), (6,33),\) and \((2,13)\) are all on the line whose equation is \(y = 5x + 3\). In other words, we need to verify the alleged equations
\[ -17 \neq 5 \cdot (-4) + 3, \quad 33 \neq 5 \cdot 6 + 3, \quad 13 \neq 5 \cdot 2 + 2. \]
Simple arithmetic shows that all three alleged equations are indeed true.
27. Find a number $t$ such that the point $(t, 2t)$ is on the line containing the points $(3, -7)$ and $(5, -15)$.

**Solution**  First we find the equation of the line containing the points $(3, -7)$ and $(5, -15)$. To do this, note that the line containing those two points has slope

$$
\frac{-7 - (-15)}{3 - 5},
$$

which equals $-4$. Thus if $(x, y)$ denotes a point on this line, then

$$
y - (-7) = -4(x - 3).
$$

Multiplying both sides of this equation by $x - 3$ and then subtracting 7 gives the equation

$$
y = -4x + 5.
$$

Now we can find a number $t$ such that the point $(t, 2t)$ is on the line given by the equation above. To do this, in the equation above replace $x$ by $t$ and $y$ by $2t$, getting

$$
2t = -4t + 5.
$$

Solving this equation for $t$, we get $t = \frac{5}{6}$.

**Check**  We should check that the three points $(3, -7)$, $(5, -15)$, and $(\frac{5}{6}, 2 \cdot \frac{5}{6})$ are all on the line whose equation is $y = -4x + 5$. In other words, we need to verify the alleged equations

$$
-7 = -4 \cdot 3 + 5, \quad -15 = -4 \cdot 5 + 5, \quad \frac{5}{3} = -4 \cdot \frac{5}{6} + 5.
$$

Simple arithmetic shows that all three alleged equations are indeed true.

29. Find the equation of the line in the $xy$-plane that contains the point $(3, 2)$ and that is parallel to the line $y = 4x - 1$.

**Solution**  The line in the $xy$-plane whose equation is $y = 4x - 1$ has slope 4. Thus each line parallel to it also has slope 4 and hence has the form

$$
y = 4x + b.
$$

for some constant $b$.

Thus we need to find a constant $b$ such that the point $(3, 2)$ is on the line given by the equation above. Replacing $x$ by 3 and replacing $y$ by 2 in the equation above, we have

$$
2 = 4 \cdot 3 + b.
$$

Solving this equation for $b$, we get $b = -10$.

Thus the line that we seek is described by the equation

$$
y = 4x - 10.
$$

31. Find the equation of the line that contains the point $(2, 3)$ and that is parallel to the line containing the points $(7, 1)$ and $(5, 6)$.

**Solution**  The line containing the points $(7, 1)$ and $(5, 6)$ has slope

$$
\frac{6 - 1}{5 - 7},
$$

which equals $\frac{5}{2}$. Thus each line parallel to it also has slope $\frac{5}{2}$ and hence has the form

$$
y = \frac{5}{2}x + b
$$

for some constant $b$.

Thus we need to find a constant $b$ such that the point $(2, 3)$ is on the line given by the equation above. Replacing $x$ by 2 and replacing $y$ by 3 in the equation above, we have

$$
3 = \frac{5}{2} \cdot 2 + b.
$$

Solving this equation for $b$, we get $b = 8$. Thus the line that we seek is described by the equation

$$
y = \frac{5}{2}x + 8.
$$

33. Find a number $t$ such that the line containing the points $(t, 2)$ and $(3, 5)$ is parallel to the line containing the points $(-1, 4)$ and $(-3, -2)$.

**Solution**  The line containing the points $(-1, 4)$ and $(-3, -2)$ has slope

$$
\frac{4 - (-2)}{-1 - (-3)},
$$

which equals 3. Thus each line parallel to it also has slope 3.

The line containing the points $(t, 2)$ and $(3, 5)$ has slope

$$
\frac{5 - 2}{3 - t},
$$

which equals $\frac{3}{5}$. From the paragraph above, we want this slope to equal 3. In other words, we need to solve the equation
\[ \frac{3}{3 - t} = 3. \]

Dividing both sides of the equation above by 3 and then multiplying both sides by \(3 - t\) gives the equation \(1 = 3 - t\). Thus \(t = 2\).

35. Find the intersection in the \(xy\)-plane of the lines \(y = 5x + 3\) and \(y = -2x + 1\).

**SOLUTION** Setting the two right sides of the equations above equal to each other, we get

\[ 5x + 3 = -2x + 1. \]

To solve this equation for \(x\), add \(2x\) to both sides and then subtract 3 from both sides, getting \(7x = -2\). Thus \(x = -\frac{2}{7}\).

To find the value of \(y\) at the intersection point, we can plug the value \(x = -\frac{2}{7}\) into either of the equations of the two lines. Choosing the first equation, we have \(y' = -5 \cdot \frac{2}{7} + 3\), which implies that \(y = \frac{11}{7}\). Thus the two lines intersect at the point \((-\frac{2}{7}, \frac{11}{7})\).

**CHECK** As a check, we can substitute the value \(x = -\frac{2}{7}\) into the equation for the second line and see if that also gives the value \(y = \frac{11}{7}\). In other words, we need to verify the alleged equation

\[ \frac{11}{7} = -2 \left(-\frac{2}{7}\right) + 1. \]

Simple arithmetic shows that this is true. Thus we indeed have the correct solution.

37. Find a number \(b\) such that the three lines in the \(xy\)-plane given by the equations \(y = 2x + b\), \(y = 3x - 5\), and \(y = -4x + 6\) have a common intersection point.

**SOLUTION** The unknown \(b\) appears in the first equation; thus our first step will be to find the point of intersection of the last two lines. To do this, we set the right sides of the last two equations equal to each other, getting

\[ 3x - 5 = -4x + 6. \]

To solve this equation for \(x\), add \(4x\) to both sides and then add 5 to both sides, getting \(7x = 11\). Thus \(x = \frac{11}{7}\). Substituting this value of \(x\) into the equation \(y = 3x - 5\), we get

\[ y = 3 \cdot \frac{11}{7} - 5. \]

Thus \(y = -\frac{2}{7}\).

At this stage, we have shown that the lines given by the equations \(y = 3x - 5\) and \(y = -4x + 6\) intersect at the point \((\frac{11}{7}, -\frac{2}{7})\). We want the line given by the equation \(y = 2x + b\) also to contain this point. Thus we set \(x = \frac{11}{7}\) and \(y = -\frac{2}{7}\) in this equation, getting

\[ -\frac{2}{7} = 2 \cdot \frac{11}{7} + b. \]

Solving this equation for \(b\), we get \(b = -\frac{24}{7}\).

**CHECK** As a check that the line given by the equation \(y = -4x + 6\) contains the point \((\frac{11}{7}, -\frac{2}{7})\), we can substitute the value \(x = \frac{11}{7}\) into the equation for that line and see if it gives the value \(y = -\frac{2}{7}\). In other words, we need to verify the alleged equation

\[ -\frac{2}{7} = -4 \cdot \frac{11}{7} + 6. \]

Simple arithmetic shows that this is true. Thus we indeed found the correct point of intersection.

We chose the line whose equation is given by \(y = -4x + 6\) for this check because the other two lines had been used in direct calculations in our solution.

39. Find the equation of the line in the \(xy\)-plane that contains the point \((4, 1)\) and that is perpendicular to the line whose equation is \(y = 3x + 5\).

**SOLUTION** The line in the \(xy\)-plane whose equation is \(y = 3x + 5\) has slope 3. Thus every line perpendicular to it has slope \(-\frac{1}{3}\). Hence the equation of the line that we seek has the form

\[ y = -\frac{1}{3}x + b \]

for some constant \(b\). We want the point \((4, 1)\) to be on this line. Substituting \(x = 4\) and \(y = 1\) into the equation above, we have

\[ 1 = -\frac{1}{3} \cdot 4 + b. \]

Solving this equation for \(b\), we get \(b = \frac{7}{3}\). Thus the equation of the line that we seek is

\[ y = -\frac{1}{3}x + \frac{7}{3}. \]
41. Find a number $t$ such that the line in the $xy$-plane containing the points $(t, 4)$ and $(2, -1)$ is perpendicular to the line $y = 6x - 7$.

SOLUTION The line in the $xy$-plane whose equation is $y = 6x - 7$ has slope 6. Thus every line perpendicular to it has slope $-\frac{1}{6}$. Thus we want the line containing the points $(t, 4)$ and $(2, -1)$ to have slope $-\frac{1}{6}$. In other words, we want
\[
\frac{4 - (-1)}{t - 2} = -\frac{1}{6}.
\]
Solving this equation for $t$, we get $t = -28$.

43. Find a number $t$ such that the line containing the points $(4, t)$ and $(-1, 6)$ is perpendicular to the line that contains the points $(3, 5)$ and $(1, -2)$.

SOLUTION The line containing the points $(3, 5)$ and $(1, -2)$ has slope
\[
\frac{5 - (-2)}{3 - 1} = \frac{7}{2},
\]
which equals $\frac{7}{2}$. Thus every line perpendicular to it has slope $-\frac{2}{7}$. Thus we want the line containing the points $(4, t)$ and $(-1, 6)$ to have slope $-\frac{2}{7}$. In other words, we want
\[
\frac{t - 6}{4 - (-1)} = -\frac{2}{7}.
\]
Solving this equation for $t$, we get $t = \frac{32}{7}$.

45. Find the midpoint of the line segment connecting $(-3, 4)$ and $(5, 7)$.

SOLUTION The midpoint of the line segment connecting $(-3, 4)$ and $(5, 7)$ is
\[
\left(\frac{-3 + 5}{2}, \frac{4 + 7}{2}\right),
\]
which equals $(1, \frac{11}{2})$.

47. Find numbers $x$ and $y$ such that $(-2, 5)$ is the midpoint of the line segment connecting $(3, 1)$ and $(x, y)$.

SOLUTION The midpoint of the line segment connecting $(3, 1)$ and $(x, y)$ is
\[
\left(\frac{3 + x}{2}, \frac{1 + y}{2}\right).
\]
We want this to equal $(-2, 5)$. Thus we must solve the equations
\[
\frac{3 + x}{2} = -2 \quad \text{and} \quad \frac{1 + y}{2} = 5.
\]
Solving these equations gives $x = -7$ and $y = 9$. 


LEARNING OBJECTIVES

By the end of this section you should be able to
- rewrite a quadratic expression using the completing-the-square technique;
- find the point on a line closest to a given point;
- derive and use the quadratic formula;
- find the center and radius of a circle from its equation;
- find the vertex of a parabola;
- recognize and work with equations of ellipses and hyperbolas.

Completing the Square

Suppose you want to find the numbers $x$ such that

$$x^2 - 5 = 0.$$  

The solution is easy to obtain. Simply add 5 to both sides of the equation above, getting $x^2 = 5$, and then conclude that $x = \pm \sqrt{5}$. Here $\pm$ indicates that we can choose either the plus sign or the minus sign.

Now consider the harder problem of finding the numbers $x$ such that

$$x^2 + 6x - 4 = 0.$$  

Nothing obvious works here. Adding either 4 or $4 - 6x$ to both sides of this equation does not produce a new equation that leads to the solution.

However, a technique called completing the square can be used to deal with equations such as the one above. The key to this technique is the identity

$$(x + t)^2 = x^2 + 2tx + t^2.$$  

The next example illustrates the technique of completing the square.

**Example 1**

Find the numbers $x$ such that

$$x^2 + 6x - 4 = 0.$$  

**Solution** The idea here is that we want the $6x$ term in the equation above to match term $2tx$ term in the expansion above for $(x + t)^2$. In other words, we want $2t = 6$, and hence we choose $t = 3$.

When $(x + 3)^2$ is expanded, we get $x^2 + 6x + 9$. The $x^2$ and $6x$ terms match the corresponding terms in the expression $x^2 + 6x$, but the expansion of $(x + 3)^2$ has an extra constant term of 9. Thus we subtract 9, rewriting $x^2 + 6x$ in the equation above as $(x + 3)^2 - 9$, getting

$$(x + 3)^2 - 9 - 4 = 0.$$  

Here $\pm$ indicates that we can choose either the plus sign or the minus sign.
Now add 13 to both sides of the equation above, getting \((x + 3)^2 = 13\), which implies that \(x + 3 = \pm \sqrt{13}\). Finally, add \(-3\) to both sides, concluding that \(x = -3 \pm \sqrt{13}\).

The general formula for the substitution to make when completing the square is shown below. Do not memorize this formula. You only need to remember that the coefficient of the \(x\) term will need to be divided by 2, and then the appropriate constant will need to be subtracted to get a correct identity.

**Completing the square**

\[
x^2 + bx = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2
\]

For example, if \(b = -10\), then the identity above becomes

\[
x^2 - 10x = (x - 5)^2 - 25.
\]

Note that the term that is subtracted is always positive because \(\left(\frac{b}{2}\right)^2\) is positive regardless of whether \(b\) is positive or negative.

The next example shows the usefulness of completing the square.

---

**EXAMPLE 2**

(a) What value of \(x\) makes \(3x^2 - 5x + 4\) as small as possible?

(b) What is the smallest value of \(3x^2 - 5x + 4\)?

**SOLUTION**

(a) We factor out the coefficient 3 from the \(x^2\) and \(x\) terms and then apply the completing-the-square identity, as follows:

\[
3x^2 - 5x + 4 = 3[x^2 - \frac{5}{3}x] + 4
\]

\[
= 3[(x - \frac{5}{6})^2 - \frac{25}{36}] + 4
\]

\[
= 3(x - \frac{5}{6})^2 - \frac{25}{12} + 4
\]

\[
= 3(x - \frac{5}{6})^2 + \frac{23}{12}
\]

The term \((x - \frac{5}{6})^2\) in the last expression is positive for all values of \(x\) except for \(x = \frac{5}{6}\), which makes \((x - \frac{5}{6})^2\) equal to 0. Thus \(3x^2 - 5x + 4\) is as small as possible when \(x = \frac{5}{6}\).

(b) We could substitute the value \(x = \frac{5}{6}\) into the expression \(3x^2 - 5x + 4\) to find the smallest value of this expression. However, without doing any work we can see from the last equation above that this expression equals \(\frac{23}{12}\) when \(x = \frac{5}{6}\). Thus \(\frac{23}{12}\) is the smallest value of \(3x^2 - 5x + 4\).
The next example illustrates another application of the completing-the-square technique.

Find the point on the line \( y = 2x - 1 \) that is closest to the point \((2, 1)\).

**SOLUTION** A typical point on the line \( y = 2x - 1 \) has coordinates \((x, 2x - 1)\). The distance between this point and \((2, 1)\) equals

\[
\sqrt{(x - 2)^2 + (2x - 1 - 1)^2},
\]

which with a bit of algebra (do it!) can be rewritten as

\[
\sqrt{5x^2 - 12x + 8}.
\]

We want to make the quantity above as small as possible, which means that we need to make \(5x^2 - 12x\) as small as possible. This can be done by completing the square:

\[
5x^2 - 12x = 5[x^2 - \frac{12}{5}x]
\]

\[
= 5[(x - \frac{6}{5})^2 - \frac{36}{25}].
\]

The last quantity will be as small as possible when \(x = \frac{6}{5}\). Plugging \(x = \frac{6}{5}\) into the equation \(y = 2x - 1\) gives \(y = \frac{7}{5}\).

Thus \((\frac{6}{5}, \frac{7}{5})\) is the point on the line \(y = 2x - 1\) that is closest to the point \((2, 1)\).

The picture in the margin shows that \((\frac{6}{5}, \frac{7}{5})\) is indeed a plausible solution.

---

**The Quadratic Formula**

**Quadratic expression**

A quadratic expression has the form

\[
ax^2 + bx + c,
\]

where \(a \neq 0\).

Consider the quadratic expression \(ax^2 + bx + c\), where \(a \neq 0\). Factor out \(a\) from the first two terms and then complete the square, as follows:

\[
ax^2 + bx + c = a \left[ x^2 + \frac{b}{a} x \right] + c
\]

\[
= a \left[ \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right] + c
\]

\[
= a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c
\]

\[
= a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}.
\]

We will follow the same pattern as in previous concrete examples and complete the square with an arbitrary quadratic expression. This will allow us to derive the quadratic formula.
Suppose now that we want to find the numbers \( x \), such that \( ax^2 + bx + c = 0 \). Setting the last expression equal to 0, we get

\[
\begin{align*}
    a \left( x + \frac{b}{2a} \right)^2 &= \frac{b^2 - 4ac}{4a},
    \\
    \text{and then dividing both sides by } a \text{ gives}
    \\
    (x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2}.
\end{align*}
\]

Regardless of the value of \( x \), the left side of the last equation is a positive number or 0. Thus if the right side is negative, the equation does not hold for any real number \( x \). In other words, if \( b^2 - 4ac < 0 \), then the equation \( ax^2 + bx + c = 0 \) has no real solutions.

If \( b^2 - 4ac \geq 0 \), then we can take the square root of both sides of the last equation, getting

\[
    x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{2a}},
\]

and then adding \( -\frac{b}{2a} \) to both sides gives

\[
    x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

By completing the square, we have derived the quadratic formula!

**Quadratic formula**

Consider the equation

\[
    ax^2 + bx + c = 0,
\]

where \( a, b, \) and \( c \) are real numbers with \( a \neq 0 \).

- If \( b^2 - 4ac < 0 \), then the equation above has no (real) solutions.
- If \( b^2 - 4ac = 0 \), then the equation above has one solution:

\[
    x = -\frac{b}{2a}.
\]

- If \( b^2 - 4ac > 0 \), then the equation above has two solutions:

\[
    x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

The quadratic formula often is useful in problems that do not initially seem to involve quadratic expressions. The following example illustrates how a simply stated problem can lead to a quadratic expression.
Find two numbers whose sum equals 7 and whose product equals 8.

**EXAMPLE 4**

Let’s call the two numbers \( s \) and \( t \). We want

\[
s + t = 7 \quad \text{and} \quad st = 8.
\]

Solving the first equation for \( s \), we have \( s = 7 - t \). Substituting this expression for \( s \) into the second equation gives \((7 - t)t = 8\), which is equivalent to the equation

\[
t^2 - 7t + 8 = 0.
\]

Using the quadratic formula to solve this equation for \( t \) gives

\[
t = \frac{7 \pm \sqrt{7^2 - 4 \cdot 8}}{2} = \frac{7 \pm \sqrt{17}}{2}.
\]

Let’s choose the solution \( t = \frac{7 + \sqrt{17}}{2} \). Plugging this value of \( t \) into the equation \( s = 7 - t \) then gives \( s = \frac{7 - \sqrt{17}}{2} \).

Thus two numbers whose sum equals 7 and whose product equals 8 are \( \frac{7 + \sqrt{17}}{2} \) and \( \frac{7 - \sqrt{17}}{2} \).

**REMARK** To check that this solution is correct, note that

\[
\frac{7 - \sqrt{17}}{2} + \frac{7 + \sqrt{17}}{2} = \frac{14}{2} = 7
\]

and

\[
\frac{7 - \sqrt{17}}{2} \cdot \frac{7 + \sqrt{17}}{2} = \frac{7^2 - \sqrt{17}^2}{4} = \frac{49 - 17}{4} = \frac{32}{4} = 8.
\]

**Circles**

The set of points that have distance 3 from the origin in a coordinate plane is the circle with radius 3 centered at the origin. The next example shows how to find an equation that describes this circle.

Find an equation that describes the circle with radius 3 centered at the origin in the \( xy \)-plane.

**EXAMPLE 5**

Recall that the distance from a point \((x, y)\) to the origin is \(\sqrt{x^2 + y^2}\). Hence a point \((x, y)\) has distance 3 from the origin if and only if

\[
\sqrt{x^2 + y^2} = 3.
\]

Squaring both sides, we get

\[
x^2 + y^2 = 9.
\]
More generally, suppose \( r \) is a positive number. Using the same reasoning as above, we see that

\[
x^2 + y^2 = r^2
\]

is the equation of the circle with radius \( r \) centered at the origin in the \( xy \)-plane.

We can also consider circles centered at points other than the origin.

**Example 6**

Find the equation of the circle in the \( xy \)-plane centered at \((2, 1)\) with radius 5.

**Solution**

This circle is the set of points whose distance from \((2, 1)\) equals 5. In other words, the circle centered at \((2, 1)\) with radius 5 is the set of points \((x, y)\) satisfying the equation

\[
\sqrt{(x - 2)^2 + (y - 1)^2} = 5.
\]

Squaring both sides, we can more conveniently describe this circle as the set of points \((x, y)\) such that

\[
(x - 2)^2 + (y - 1)^2 = 25.
\]

Using the same reasoning as in the example above, we get the following more general result:

**Equation of a circle**

The circle with center \((h, k)\) and radius \( r \) is the set of points \((x, y)\) satisfying the equation

\[
(x - h)^2 + (y - k)^2 = r^2.
\]

Sometimes the equation of a circle may be in a form in which the radius and center are not obvious. You may then need to complete the square to find the radius and center. The following example illustrates this procedure:

**Example 7**

Find the radius and center of the circle in the \( xy \)-plane described by

\[
x^2 + 4x + y^2 - 6y = 12.
\]

**Solution**

Completing the square, we have

\[
12 = x^2 + 4x + y^2 - 6y
\]

\[
= (x + 2)^2 - 4 + (y - 3)^2 - 9
\]

\[
= (x + 2)^2 + (y - 3)^2 - 13.
\]

Adding 13 to the first and last sides of the equation above shows that

\[
(x + 2)^2 + (y - 3)^2 = 25.
\]

Thus we have a circle with radius 5 centered at \((-2, 3)\).
Ellipses

**Ellipse**

Stretching a circle horizontally and/or vertically produces a curve called an **ellipse**.

Find an equation describing the ellipse in the $xy$-plane produced by stretching a circle of radius 1 centered at the origin horizontally by a factor of 5 and vertically by a factor of 3.

**EXAMPLE 8**

To find an equation describing this ellipse, consider a typical point $(u, v)$ on the circle of radius 1 centered at the origin. Thus $u^2 + v^2 = 1$.

![Graph showing a circle and an ellipse]

**Stretching horizontally by a factor of 5 and vertically by a factor of 3 transforms the circle on the left into the ellipse on the right.**

Stretching horizontally by a factor of 5 and stretching vertically by a factor of 3 transforms the point $(u, v)$ to the point $(5u, 3v)$. Rewrite the equation $u^2 + v^2 = 1$ in terms of this new point, getting

$$
\frac{(5u)^2}{25} + \frac{(3v)^2}{9} = 1.
$$

Write the transformed point $(5u, 3v)$ as $(x, y)$, thus setting $x = 5u$ and $y = 3v$, getting

$$
\frac{x^2}{25} + \frac{y^2}{9} = 1,
$$

which is the equation of the ellipse shown above on the right.

The points $(\pm 5, 0)$ and $(0, \pm 3)$ satisfy this equation and thus lie on the ellipse, as can also be seen from the figure above.

More generally, suppose $a$ and $b$ are positive numbers. Suppose the circle of radius 1 centered at the origin is stretched horizontally by a factor of $a$ and stretched vertically by a factor of $b$. Using the same reasoning as above (just replace 5 by $a$ and replace 3 by $b$), we see that the equation of the resulting ellipse in the $xy$-plane is

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
$$

The German mathematician Johannes Kepler, who in 1609 published his discovery that the orbits of the planets are ellipses, not circles or combinations of circles as had been previously thought.

Viewing an ellipse as a stretched circle will lead us to the formula for the area inside an ellipse in Section 2.4.
The points \((\pm a, 0)\) and \((0, \pm b)\) satisfy this equation and lie on the ellipse.

Planets have orbits that are ellipses, but you may be surprised to learn that the sun is not located at the center of those orbits. Instead, the sun is located at what is called a **focus** of each elliptical orbit. The plural of focus is **foci**, which are defined as follows:

### Foci of an ellipse

The foci of an ellipse are two points with the property that the sum of the distances from the foci to a point on the ellipse is a constant independent of the point on the ellipse.

As we will see in the next example, the foci for the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \) are the points \((-4, 0)\) and \((4, 0)\). This example shows how to verify that a pair of points are foci for an ellipse.

---

**Example 9**

(a) Find a formula in terms of \(x\) for the distance from a typical point \((x, y)\) on the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \) to the point \((4, 0)\).

(b) Find a formula in terms of \(x\) for the distance from a typical point \((x, y)\) on the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \) to the point \((-4, 0)\).

(c) Show that \((4, 0)\) and \((-4, 0)\) are foci of the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \).

**Solution**

(a) The distance from \((x, y)\) to the point \((4, 0)\) is \(\sqrt{(x - 4)^2 + y^2}\), which equals

\[\sqrt{x^2 - 8x + 16 + y^2}.\]

We want an answer solely in terms of \(x\), assuming that \((x, y)\) lies on the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \). Solving the ellipse equation for \(y^2\), we have

\[y^2 = 9 - \frac{9}{25}x^2.\]

Substituting this expression for \(y^2\) into the expression above shows that the distance from \((x, y)\) to \((4, 0)\) equals

\[\sqrt{25 - 8x + \frac{16}{25}x^2},\]

which equals \(\sqrt{(5 - \frac{4}{5}x)^2}\), which equals \(5 - \frac{4}{5}x\).

(b) The distance from \((x, y)\) to the point \((-4, 0)\) is \(\sqrt{(x + 4)^2 + y^2}\), which equals

\[\sqrt{x^2 + 8x + 16 + y^2}.\]

Now proceed as in the solution to part (a), with \(-8x\) replaced by \(8x\), concluding that the distance from \((x, y)\) to \((-4, 0)\) is \(5 + \frac{4}{5}x\).
(c) As we know from the previous two parts of this example, if \((x, y)\) is a point on the ellipse \(\frac{x^2}{25} + \frac{y^2}{9} = 1\), then

\[
\text{distance from } (x, y) \text{ to } (4, 0) \text{ is } 5 - \frac{4}{5}x
\]

and

\[
\text{distance from } (x, y) \text{ to } (-4, 0) \text{ is } 5 + \frac{4}{5}x.
\]

Adding these distances, we see that the sum equals 10, which is a constant independent of the point \((x, y)\) on the ellipse. Thus \((4, 0)\) and \((-4, 0)\) are foci of this ellipse.

The next result generalizes the example above. The verification of this result is outlined in Problems 72–77. To do this verification, simply use the ideas from the example above.

**Formula for the foci of an ellipse**

- If \(a > b > 0\), then the foci of the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) are the points \((\sqrt{a^2 - b^2}, 0)\) and \((-\sqrt{a^2 - b^2}, 0)\).

- If \(b > a > 0\), then the foci of the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) are the points \((0, \sqrt{b^2 - a^2})\) and \((0, -\sqrt{b^2 - a^2})\).

**Parabolas**

A kicked or thrown football follows a path shaped like the curve in the margin. You may be tempted to think that this curve is half of an ellipse. However, the equations of gravity show that objects thrown into the air follow a path that is part of a parabola, not part of an ellipse. Thus we now turn our attention to parabolas and the equations that define them.

Parabolas can be defined geometrically, but for our purposes it is simpler to look at a class of parabolas that are easy to define algebraically. For now, we will restrict our attention to parabolas in the \(xy\)-plane that have a vertical line of symmetry:
Chapter 2  Combining Algebra and Geometry

Parabolas

The graph of an equation of the form

\[ y = ax^2 + bx + c, \]

where \( a \neq 0 \), is called a parabola.

For example, the graph of the equation \( y = x^2 \) is the familiar parabola shown in the margin. This parabola is symmetric about the \( y \)-axis, meaning that the parabola is unchanged if it is flipped across the \( y \)-axis. Note that this line of symmetry intersects this parabola at the origin, which is the lowest point on this parabola.

So that you can become familiar with the shape of parabolas, the figure below shows two more typical parabolas.

The parabola on the left is symmetric about the line \( x = -3 \) (shown in red), which intersects the parabola at its lowest point. The parabola on the right is symmetric about the line \( x = \frac{5}{6} \) (shown in red), which intersects the parabola at its highest point.

Every parabola is symmetric about some line. The point where this line of symmetry intersects the parabola is sufficiently important to deserve its own name:

Vertex

The vertex of a parabola is the point where the line of symmetry of the parabola intersects the parabola.

For example, the figure above shows that the vertex of the parabola \( y = x^2 + 6x + 11 \) on the left is \((-3, 2)\), which is the lowest point on the graph. The figure above also shows that the vertex of the parabola \( y = -3x^2 + 5x + 1 \) is \((-\frac{3}{6}, \frac{37}{12})\), which is the highest point on the graph.

The parabolas above exhibit the typical behavior described by the following result:
**Parabola direction**

Consider the parabola given by the equation

\[ y = ax^2 + bx + c, \]

where \( a \neq 0 \).

- If \( a > 0 \), then the parabola opens upward and the vertex of the parabola is the lowest point on the graph.
- If \( a < 0 \), then the parabola opens downward and the vertex of the parabola is the highest point on the graph.

When we deal with function transformations in Section 3.2, we will see why the result above holds and we will learn how to use function transformations to obtain the graph of a parabola by appropriate transformations of the graph of \( y = x^2 \). Thus for now we give only one example showing how to find the vertex of a parabola.

---

**Example 10**

(a) For what value of \( x \) does \( x^2 + 6x + 11 \) attain its minimum value?

(b) Find the vertex of the graph of \( y = x^2 + 6x + 11 \).

**Solution**

(a) First complete the square to rewrite \( x^2 + 6x + 11 \), as follows:

\[
\begin{align*}
x^2 + 6x + 11 &= (x + 3)^2 - 9 + 11 \\
&= (x + 3)^2 + 2
\end{align*}
\]

The last expression shows that \( x^2 + 6x + 11 \) takes on its minimum value when \( x = -3 \), because \((x + 3)^2\) is positive for all values of \( x \) except \( x = -3 \).

(b) The last expression above shows that \( x^2 + 6x + 11 \) equals 2 when \( x = -3 \). Thus the vertex of the graph of \( y = x^2 + 6x + 11 \) is the point \((-3, 2)\), as shown in the graph of this parabola above.

---

**Hyperbolas**

Some comets and all planets travel in orbits that are ellipses. However, many comets have orbits that are not ellipses but instead lie on hyperbolas, which is another category of curves. Thus we now turn our attention to hyperbolas and the equations that define them.

Hyperbolas can be defined geometrically, but we will restrict attention to hyperbolas in the \( xy \)-plane that can be defined in the following simple fashion:

\[
\text{A comet whose orbit lies on a hyperbola will come near earth at most once. A comet whose orbit is an ellipse will return periodically.}
\]
Hyperbolas

The graph of an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = c,$$

where $a$, $b$, and $c$ are nonzero numbers, is called a hyperbola.

The figure below shows the graph of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ in blue for $-6 \leq x \leq 6$. Note that this hyperbola consists of two branches rather than the single curve we get for ellipses and parabolas. Some key properties of this graph, which are typical of hyperbolas, are discussed in the next example.

**Example 11**

Consider the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$.

(a) Explain why this hyperbola intersects the $y$-axis at $(0, 4)$ and $(0, -4)$.
(b) Explain why the hyperbola contains no points $(x, y)$ with $|y| < 4$.
(c) Explain why points $(x, y)$ on the hyperbola for large values of $x$ are near the line $y = \frac{4}{3}x$ or the line $y = -\frac{4}{3}x$.

**Solution**

(a) If $x = 0$, then the equation defining the hyperbola shows that $\frac{y^2}{16} = 1$. Thus $y = \pm 4$. Hence the hyperbola intersects the $y$-axis, which is defined by $x = 0$, at $(0, 4)$ and $(0, -4)$.

(b) If $(x, y)$ is a point on the hyperbola, then

$$y^2 = 16\left(1 + \frac{x^2}{9}\right) \geq 16,$$

which implies that $|y| \geq 4$. Thus the hyperbola contains no points $(x, y)$ with $|y| < 4$.

(c) The equation defining this parabola can be rewritten in the form

$$\frac{y^2}{16} = \frac{16}{9} + \frac{16}{x^2}$$

if $x \neq 0$. If $x$ is large, then the equation above implies that

$$\frac{y^2}{x^2} \approx \frac{16}{9}.$$

Taking square roots then shows that $y \approx \pm \frac{4}{3}x$.

Each branch of a hyperbola may appear to be shaped like a parabola, but these curves are not parabolas. As one notable difference, a parabola cannot have the behavior described in part (c) of the example above.

Compare the following definition of the foci of a hyperbola with the definition earlier in this section of the foci of an ellipse.
Foci of a hyperbola

The foci of a hyperbola are two points with the property that the difference of the distances from the foci to a point on the hyperbola is a constant independent of the point on the hyperbola.

As we will see in the next example, the foci for the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ are the points $(0, -5)$ and $(0, 5)$. This example shows how to verify that a pair of points are foci for a hyperbola.

(a) Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ with $y > 0$ on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ to the point $(0, -5)$.

(b) Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ with $y > 0$ on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ to the point $(0, 5)$.

(c) Show that $(0, -5)$ and $(0, 5)$ are foci of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$.

SOLUTION

(a) The distance from $(x, y)$ to the point $(0, -5)$ is $\sqrt{x^2 + (y + 5)^2}$, which equals $\sqrt{x^2 + y^2 + 10y + 25}$.

We want an answer solely in terms of $y$, assuming that $(x, y)$ lies on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ and $y > 0$. Solving the hyperbola equation for $x^2$, we have

$$x^2 = \frac{9}{16}y^2 - 9.$$ 

Substituting this expression for $x^2$ into the expression above shows that the distance from $(x, y)$ to $(0, -5)$ equals

$$\sqrt{\frac{25}{16}y^2 + 10y + 16},$$

which equals $\sqrt{\left(\frac{5}{4}y + 4\right)^2}$, which equals $\frac{5}{4}y + 4$.

(b) The distance from $(x, y)$ to the point $(0, 5)$ is $\sqrt{x^2 + (y - 5)^2}$, which equals $\sqrt{x^2 + y^2 - 10y + 25}$.

Now proceed as in the solution to part (a), with $10y$ replaced by $-10y$, concluding that the distance from $(x, y)$ to $(0, 5)$ is $\frac{5}{4}y - 4$.

(c) As we know from the previous two parts of this example, if $(x, y)$ is a point on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ and $y > 0$, then

- distance from $(x, y)$ to $(0, -5)$ is $\frac{5}{4}y + 4$
- distance from $(x, y)$ to $(0, 5)$ is $\frac{5}{4}y - 4$.

Subtracting these distances, we see that the difference equals 8, which is a constant independent of the point $(x, y)$ on the hyperbola. Thus $(0, -5)$ and $(0, 5)$ are foci of this hyperbola.

PROBLEMS 81–84 show why the graph of $y = \frac{1}{x}$ is also called a hyperbola.
Isaac Newton showed that a comet’s orbit around a star lies either on an ellipse or on a parabola (rare) or on a hyperbola with the star at one of the foci. For example, if units are chosen so that the orbit of a comet is the upper branch of the hyperbola  \( \frac{y^2}{16} - \frac{x^2}{9} = 1 \), then the star must be located at \((0, 5)\).

The next result generalizes the example above. The verification of this result is outlined in Problems 78–80. To do this verification, simply use the ideas from the example above.

**Formula for the foci of a hyperbola**

If \( a \) and \( b \) are nonzero numbers, then the foci of the hyperbola \( \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \) are the points \( (0, -\sqrt{a^2 + b^2}) \) and \( (0, \sqrt{a^2 + b^2}) \).

Sometimes when a new comet is discovered, there are not enough observations to determine whether the comet is in an elliptical orbit or in a hyperbolic orbit. The distinction is important, because a comet in a hyperbolic orbit will disappear and never again be visible from earth.

**EXERCISES**

For Exercises 1–12, use the following information: If an object is thrown straight up into the air from height \( H \) feet at time 0 with initial velocity \( V \) feet per second, then at time \( t \) seconds the height of the object is

\[-16.1t^2 + Vt + H\]

feet. This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

Some notebook computers have a sensor that detects sudden changes in motion and stops the notebook’s hard drive, protecting it from damage.

1. Suppose a notebook computer is accidentally knocked off a shelf that is six feet high. How long before the computer hits the ground?

2. Suppose a notebook computer is accidentally knocked off a desk that is three feet high. How long before the computer hits the ground?

3. Suppose the motion detection/protection mechanism of a notebook computer takes 0.3 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

4. Suppose the motion detection/protection mechanism of a notebook computer takes 0.4 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

5. Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.

   (a) How long before the ball hits the ground?

   (b) How long before the ball reaches its maximum height?

   (c) What is the ball’s maximum height?

6. Suppose a ball is tossed straight up into the air from height 4 feet with initial velocity 40 feet per second.

   (a) How long before the ball hits the ground?

   (b) How long before the ball reaches its maximum height?

   (c) What is the ball’s maximum height?

7. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?
8. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball stay in the air for 3 seconds?

9. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach its maximum height after 2 seconds?

10. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach its maximum height after 2 seconds?

11. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach a height of 50 feet?

12. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach a height of 70 feet?

13. Find the equation of the circle in the xy-plane centered at (3, -2) with radius 7.

14. Find the equation of the circle in the xy-plane centered at (-4, 5) with radius 6.

15. Find two choices for b such that (5, b) is on the circle with radius 4 centered at (3, 6).

16. Find two choices for b such that (b, 4) is on the circle with radius 3 centered at (-1, 6).

17. Find all numbers x such that
\[ \frac{x - 1}{x + 3} = \frac{2x - 1}{x + 2}. \]

18. Find all numbers x such that
\[ \frac{3x + 2}{x - 2} = \frac{2x - 1}{x - 1}. \]

19. Find two numbers w such that the points (3, 1), (w, 4), and (5, w) all lie on a straight line.

20. Find two numbers r such that the points (-1, 4), (r, 2r), and (1, r) all lie on a straight line.

21. The graph of the equation
\[ x^2 - 6x + y^2 + 8y = -25 \]
contains exactly one point. Find the coordinates of that point.

22. The graph of the equation
\[ x^2 + 5x + y^2 - 3y = -\frac{17}{2} \]
contains exactly one point. Find the coordinates of that point.

23. Find the point on the line y = 3x + 1 in the xy-plane that is closest to the point (2, 4).

24. Find the point on the line y = 2x - 3 in the xy-plane that is closest to the point (5, 1).

25. Find a number t such that the distance between (2, 3) and (t, 2t) is as small as possible.

26. Find a number t such that the distance between (-2, 1) and (3t, 2t) is as small as possible.

27. Find the length of the graph of the curve defined by
\[ y = \sqrt{9 - x^2} \]
with -3 \leq x \leq 3.

28. Find the length of the graph of the curve defined by
\[ y = \sqrt{25 - x^2} \]
with 0 \leq x \leq 5.

29. Find the two points where the circle of radius 2 centered at the origin intersects the circle of radius 3 centered at (3, 0).

30. Find the two points where the circle of radius 3 centered at the origin intersects the circle of radius 4 centered at (5, 0).

31. Find the equation of the circle in the xy-plane centered at the origin with circumference 9.

32. Find the equation of the circle in the xy-plane centered at (3, 7) with circumference 5.

33. Find the equation of the circle centered at the origin in the uv-plane that has twice the circumference of the circle whose equation equals
\[ u^2 + v^2 = 10. \]

34. Find the equation of the circle centered at the origin in the tw-plane that has three times the circumference of the circle whose equation equals
\[ t^2 + w^2 = 5. \]

For Exercises 35 and 36, find the following information about the circles in the xy-plane described by the given equation:
35. \( x^2 - 8x + y^2 + 2y = -14 \)
36. \( x^2 + 5x + y^2 - 6y = 3 \)
37. Find the intersection of the line containing the points \((2, 3)\) and \((4, 7)\) and the circle with radius \(\sqrt{5}\) centered at \((3, -3)\).
38. Find the intersection of the line containing the points \((3, 4)\) and \((1, 8)\) and the circle with radius \(\sqrt{3}\) centered at \((2, 9)\).

For Exercises 39–44, find the vertex of the graph of the given equation.

39. \( y = 7x^2 - 12 \)
40. \( y = -9x^2 - 5 \)
41. \( y = (x - 2)^2 - 3 \)
42. \( y = (x + 3)^2 + 4 \)
43. \( y = (2x - 5)^2 + 6 \)
44. \( y = (7x + 3)^2 + 5 \)

In Exercises 45–48, for the given equation:

(a) Write the right side of the equation in the form \(k(x + t)^2 + r\).

(b) Find the value of \(x\) where the right side of the equation attains its minimum value or its maximum value.

(c) Find the minimum or maximum value of the right side of the equation.

(d) Find the vertex of the graph of the equation.

63. Suppose \(at^2 + 5t + 4 > 0\) for every real number \(t\). Show that \(a > \frac{25}{16}\).
64. Suppose \(a \neq 0\) and \(b^2 \geq 4ac\). Verify by direct substitution that if
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \]
then \(ax^2 + bx + c = 0\).
65. Suppose \(a \neq 0\) and \(b^2 \geq 4ac\). Verify by direct calculation that
\[ ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right). \]
66. Find a number $\phi$ such that in the figure below, the yellow rectangle is similar to the large rectangle formed by the union of the blue square and the yellow rectangle.

[Diagram showing rectangles and the golden ratio]

The number $\phi$ that solves this problem is called the golden ratio (the symbol $\phi$ is the Greek letter phi). Rectangles whose ratio between the length of the long side and the length of the short side equals $\phi$ are supposedly the most aesthetically pleasing rectangles. The large rectangle formed by the union of the blue square and the yellow rectangle has the golden ratio, as does the yellow rectangle. Many works of art feature rectangles with the golden ratio.

67. Suppose $a$, $b$, and $c$ are numbers with $a \neq 0$. Show that the vertex of the graph of

$$y = ax^2 + bx + c$$

is the point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

68. Suppose $a$, $b$, and $c$ are numbers with $a \neq 0$ and that only one number $t$ satisfies the equation

$$at^2 + bt + c = 0.$$ 

Show that $t$ is the first coordinate of the vertex of the graph of $y = ax^2 + bx + c$ and that the second coordinate of the vertex equals 0.

69. Suppose $a$, $b$, and $c$ are numbers such that exactly two real numbers $t$ satisfy the equation $at^2 + bt + c = 0$. Show that the average of these two real numbers is the first coordinate of the vertex of the graph of $y = ax^2 + bx + c$.

70. Suppose $b$ and $c$ are numbers such that the equation

$$x^2 + bx + c = 0$$

has no real solutions. Explain why the equation

$$x^2 + bx - c = 0$$

has two real solutions.

71. Show that there do not exist two real numbers whose sum is 7 and whose product is 13.

72. Suppose $a > b > 0$. Find a formula in terms of $x$ for the distance from a typical point $(x, y')$ on the ellipse $\frac{x^2}{a^2} + \frac{y'^2}{b^2} = 1$ to the point $(\sqrt{a^2-b^2}, 0)$.

73. Suppose $a > b > 0$. Find a formula in terms of $x$ for the distance from a typical point $(x, y')$ on the ellipse $\frac{x^2}{a^2} + \frac{y'^2}{b^2} = 1$ to the point $(-\sqrt{a^2-b^2}, 0)$.

74. Suppose $a > b > 0$. Use the results of the two previous problems to show that $(-\sqrt{a^2-b^2}, 0)$ and $(\sqrt{a^2-b^2}, 0)$ are foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ 

75. Suppose $b > a > 0$. Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, \sqrt{b^2-a^2})$.

76. Suppose $b > a > 0$. Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, -\sqrt{b^2-a^2})$.

77. Suppose $b > a > 0$. Use the results of the two previous problems to show that $(0, \sqrt{b^2-a^2})$ and $(0, -\sqrt{b^2-a^2})$ are foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ 

78. Suppose $a$ and $b$ are nonzero numbers. Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ with $y > 0$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the point $(0, -\sqrt{a^2+b^2})$.

79. Suppose $a$ and $b$ are nonzero numbers. Find a formula in terms of $y$ for the distance from a typical point $(x, y)$ with $y > 0$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the point $(0, \sqrt{a^2+b^2})$.

80. Suppose $a$ and $b$ are nonzero numbers. Use the results of the two previous problems to show that $(0, -\sqrt{a^2+b^2})$ and $(0, \sqrt{a^2+b^2})$ are foci of the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$ 

81. Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to the point $(-\sqrt{2}, -\sqrt{2})$ is $x + \frac{1}{x} + \sqrt{2}$. [See Example 7 in Section 4.1 for a graph of $y = \frac{1}{x}$]
82. Suppose \( x > 0 \). Show that the distance from \((x, \frac{1}{x})\) to the point \((\sqrt{2}, \sqrt{2})\) is \( x + \frac{1}{x} - \sqrt{2} \).

83. Suppose \( x > 0 \). Show that the distance from \((x, \frac{1}{x})\) to \((-\sqrt{2}, -\sqrt{2})\) minus the distance from \((x, \frac{1}{x})\) to \((\sqrt{2}, \sqrt{2})\) equals \( 2\sqrt{2} \).

84. Explain why the result of the previous problem justifies calling the curve \( y = \frac{1}{x} \) a hyperbola with foci at \((-\sqrt{2}, -\sqrt{2})\) and \((\sqrt{2}, \sqrt{2})\).

85. Explain why \( \text{graph } x^2/25 + y^2/9 = 1 \) in Wolfram|Alpha produces what appears to be a circle rather than a typical ellipse-shape as shown in Examples 8 and 9. [Hint: Notice the scales on the two axes.]

WORKED-OUT SOLUTIONS to Odd-numbered Exercises

For Exercises 1–12, use the following information:

If an object is thrown straight up into the air from height \( H \) feet at time 0 with initial velocity \( V \) feet per second, then at time \( t \) seconds the height of the object is

\[-16.1t^2 + Vt + H\]

feet. This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

Some notebook computers have a sensor that detects sudden changes in motion and stops the notebook's hard drive, protecting it from damage.

1. Suppose a notebook computer is accidentally knocked off a shelf that is six feet high. How long before the computer hits the ground?

**SOLUTION** In this case, we have \( V = 0 \) and \( H = 6 \) in the formula above. We want to know the time \( t \) at which the height equals 0. In other words, we need to solve the equation

\[-16.1t^2 + 6 = 0.\]

The solutions to the equation are \( t = \pm \sqrt{\frac{6}{16.1}} \).

The negative value makes no sense here, and thus \( t = \sqrt{\frac{6}{16.1}} \approx 0.61 \) seconds.

3. Suppose the motion detection/protection mechanism of a notebook computer takes 0.3 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

**SOLUTION** We want to find the initial height \( H \) such that the notebook computer will hit the ground (meaning have height 0) after 0.3 seconds. In other words, we want

\[-16.1 \times 0.3^2 + H = 0.\]

Thus \( H = 16.1 \times 0.3^2 = 1.449 \) feet.

5. Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.

(a) How long before the ball hits the ground?

The quadratic formula shows that \( t \approx 1.46 \) seconds (the other solution produced by the quadratic formula has been discarded because it is negative).

(b) Completing the square, we have

\[-16.1t^2 + 20t + 5\]

\[= -16.1 \left[ t^2 - \frac{20}{16.1} t \right] + 5\]

\[= -16.1 \left[ \left( t - \frac{10}{16.1} \right)^2 - \left( \frac{10}{16.1} \right)^2 \right] + 5\]

\[= -16.1 \left( t - \frac{10}{16.1} \right)^2 + \frac{100}{16.1} + 5.\]

The expression above gives the height of the ball at time \( t \). Thus the ball reaches its maximum height when \( t = \frac{10}{16.1} \approx 0.62 \) seconds.

(c) The solution to part (b) shows that the maximum height of the ball is \( \frac{100}{16.1} + 5 \approx 11.2 \) feet.
7. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?

**SOLUTION** Suppose the initial velocity of the ball is $V$. Then the height of the ball at time $t$ is

$$-16.1t^2 + Vt + 5.$$

We want $t = 4$ when the ball hits the ground, meaning its height is 0. In other words, we want

$$-16.1 \times 4^2 + 4V + 5 = 0.$$

Solving this equation for $V$ gives $V = 63.15$ feet per second.

9. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach its maximum height after 1 second?

**SOLUTION** Suppose the initial velocity of the ball is $V$. Then the height of the ball at time $t$ is

$$-16.1t^2 + Vt + 5.$$

We want $Vt$ to have the ball reach a height of 50 feet.

We want 1

We want $t = 1$. In other words, we want $1 - \frac{V}{32.2} = 0$, which implies that $V = 32.2$ feet per second.

11. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach a height of 50 feet?

**SOLUTION** Suppose the initial velocity of the ball is $V$. As can be seen from the solution to Exercise 9, the maximum height of the ball is $\frac{V^2}{64.4} + 5$, which we want to equal 50. Solving the equation

$$\frac{V^2}{64.4} + 5 = 50$$

for $V$ gives $V \approx 53.83$ feet per second.

13. Find the equation of the circle in the $xy$-plane centered at $(3, -2)$ with radius 7.

**SOLUTION** The equation of this circle is

$$(x - 3)^2 + (y + 2)^2 = 49.$$
21. The graph of the equation 
\[ x^2 - 6x + y^2 + 8y = -25 \]
contains exactly one point. Find the coordinates of that point.

**SOLUTION** We complete the square to rewrite the equation above:

\[-25 = x^2 - 6x + y^2 + 8y \\
= (x - 3)^2 - 9 + (y + 4)^2 - 16 \\
= (x - 3)^2 + (y + 4)^2 - 25.\]

Thus the original equation can be rewritten as 
\[(x - 3)^2 + (y + 4)^2 = 0.\] The only solution to this equation is \(x = 3\) and \(y = -4\). Thus \((3, -4)\) is the only point on the graph of this equation.

23. Find the point on the line \(y = 3x + 1\) in the \(xy\)-plane that is closest to the point \((2, 4)\).

**SOLUTION** A typical point on the line \(y = 3x + 1\) in the \(xy\)-plane has coordinates \((x, 3x + 1)\). The distance between this point and \((2, 4)\) equals

\[ \sqrt{(x - 2)^2 + (3x + 1 - 4)^2}, \]

which with a bit of algebra can be rewritten as 
\[ \sqrt{10x^2 - 22x + 13}. \]

We want to make the quantity above as small as possible, which means that we need to make \(10x^2 - 22x\) as small as possible. This can be done by completing the square:

\[ 10x^2 - 22x = 10\left[x^2 - \frac{11}{5}x\right] \\
= 10\left(x - \frac{11}{10}\right)^2 - 12\frac{1}{100}. \]

The last quantity will be as small as possible when \(x = \frac{11}{10}\). Plugging \(x = \frac{11}{10}\) into the equation \(y = 3x + 1\) gives \(y = \frac{43}{10}\). Thus \((\frac{11}{10}, \frac{43}{10})\) is the point on the line \(y = 3x + 1\) that is closest to the point \((2, 4)\).

25. Find a number \(t\) such that the distance between \((2, 3)\) and \((t, 2t)\) is as small as possible.

**SOLUTION** The distance between \((2, 3)\) and \((t, 2t)\) equals

\[ \sqrt{(t - 2)^2 + (2t - 3)^2}.\]

We want to make this as small as possible, which happens when 
\[(t - 2)^2 + (2t - 3)^2 \]

is as small as possible. Note that 
\[(t - 2)^2 + (2t - 3)^2 = 5t^2 - 16t + 13.\]

This will be as small as possible when \(5t^2 - 16t\) is as small as possible. To find when that happens, we complete the square:

\[ 5t^2 - 16t = 5\left(t^2 - \frac{16}{5}t\right) \]
\[= 5\left(t - \frac{8}{5}\right)^2 - \frac{64}{25}.\]

This quantity is made smallest when \(t = \frac{8}{5}\).

27. Find the length of the graph of the curve defined by 
\[ y = \sqrt{9 - x^2} \]
with \(-3 \leq x \leq 3\).

**SOLUTION** Squaring both sides of the equation \(y = \sqrt{9 - x^2}\) and then adding \(x^2\) to both sides gives the equation \(x^2 + y^2 = 9\), which is the equation of the circle of radius 3 centered at the origin. However, the equation \(y = \sqrt{9 - x^2}\) implies that \(y \geq 0\), and thus we have only the top half of the circle.

The entire circle of radius 3 has circumference \(6\pi\). Thus the graph of \(y = \sqrt{9 - x^2}\), which is half of the circle, has length \(3\pi\).

29. Find the two points where the circle of radius 2 centered at the origin intersects the circle of radius 3 centered at \((3, 0)\).

**SOLUTION** The equations of these two circles are 
\[ x^2 + y^2 = 4 \quad \text{and} \quad (x - 3)^2 + y^2 = 9. \]

Subtracting the first equation from the second equation, we get 
\[(x - 3)^2 - x^2 = 5,\]
which simplifies to the equation \(-6x + 9 = 5\), whose solution is \(x = \frac{2}{3}\). Plugging this value of \(x\) into either of the equations above and solving for \(y\) gives \(y = \pm \frac{\sqrt{6}}{3}\). Thus the two circles intersect at the points \((\frac{2}{3}, \frac{\sqrt{6}}{3})\) and \((\frac{2}{3}, -\frac{\sqrt{6}}{3})\).
31. Find the equation of the circle in the $xy$-plane centered at the origin with circumference 9.

**SOLUTION** Let $r$ denote the radius of this circle. Then $2\pi r = 9$, which implies that $r = \frac{9}{2\pi}$. Thus the equation of the circle is

$$x^2 + y^2 = \frac{81}{4\pi^2}.$$ 

33. Find the equation of the circle centered at the origin in the $uv$-plane that has twice the circumference of the circle whose equation equals $u^2 + v^2 = 10$.

**SOLUTION** The equation given above describes a circle centered at the origin whose radius equals $\sqrt{10}$. Because the circumference is proportional to the radius, if we want a circle with twice the circumference then we need to double the radius. Thus the circle we seek has radius $2\sqrt{10}$. Because $(2\sqrt{10})^2 = 2^2 \cdot \sqrt{10}^2 = 40$, the equation we seek is

$$u^2 + v^2 = 40.$$ 

37. Find the intersection of the line containing the points $(2, 3)$ and $(4, 7)$ and the circle with radius $\sqrt{15}$ centered at $(3, -3)$.

**SOLUTION** First we find the equation of the line containing the points $(2, 3)$ and $(4, 7)$. This line will have slope $\frac{7-3}{4-2}$, which equals 2. Thus the equation of this line will have the form $y = 2x + b$. Because $(2, 3)$ is on this line, we can substitute $x = 2$ and $y = 3$ into the last equation and then solve for $b$, getting $b = -1$. Thus the equation of the line containing the points $(2, 3)$ and $(4, 7)$ is

$$y = 2x - 1.$$ 

The equation of the circle with radius $\sqrt{15}$ centered at $(3, -3)$ is

$$(x - 3)^2 + (y + 3)^2 = 15.$$ 

To find the intersection of the circle and the line, we replace $y$ by $2x - 1$ in the equation above, getting

$$(x - 3)^2 + (2x + 2)^2 = 15.$$ 

Expanding the terms in the equation above and then collecting terms gives the equation

$$5x^2 + 2x - 2 = 0.$$ 

Using the quadratic formula, we then find that

$$x = \frac{-1 + \sqrt{17}}{5} \quad \text{or} \quad x = \frac{-1 - \sqrt{17}}{5}.$$ 

Substituting these values of $x$ into the equation $y = 2x - 1$ shows that the line intersects the circle in the points

$$\left(\frac{-1 + \sqrt{17}}{5}, -7 + 2\sqrt{17}\right)$$
and
$$\left(\frac{-1 - \sqrt{17}}{5}, -7 - 2\sqrt{17}\right).$$

For Exercises 35 and 36, find the following information about the circles in the $xy$-plane described by the given equation:

(a) center (b) radius (c) diameter (d) circumference

35. $x^2 - 8x + y^2 + 2y = -14$

**SOLUTION** Completing the square, we can rewrite the left side of this equation as follows:

$$x^2 - 8x + y^2 + 2y = (x - 4)^2 - 16 + (y + 1)^2 - 1 = (x - 4)^2 + (y + 1)^2 - 17.$$ 

Substituting this expression into the left side of the original equation and then adding 17 to both sides shows that the original equation is equivalent to the equation

$$(x - 4)^2 + (y + 1)^2 = 3.$$ 

(a) The equation above shows that this circle has center $(4, -1)$.

(b) The equation above shows that this circle has radius $\sqrt{3}$.
For Exercises 39–44, find the vertex of the graph of the given equation.

39. \( y = 7x^2 - 12 \)

**SOLUTION** The value of \( 7x^2 - 12 \) is minimized when \( x = 0 \). When \( x = 0 \), the value of \( 7x^2 - 12 \) equals \(-12\). Thus the vertex of the graph of \( y = 7x^2 - 12 \) is \((0, -12)\).

41. \( y = (x - 2)^2 - 3 \)

**SOLUTION** The value of \((x - 2)^2 - 3\) is minimized when \( x = 2 \). When \( x = 2 \), the value of \((x - 2)^2 - 3\) equals \(-3\). Thus the vertex of the graph of \( y = (x - 2)^2 - 3 \) is \((2, -3)\).

43. \( y = (2x - 5)^2 + 6 \)

**SOLUTION** The value of \((2x - 5)^2 + 6\) is minimized when \( 2x - 5 = 0 \). This happens when \( x = \frac{5}{2} \), which makes the value of \((2x - 5)^2 + 6\) equal to 6. Thus the vertex of the graph of \( y = (2x - 5)^2 + 6 \) is \((\frac{5}{2}, 6)\).

In Exercises 45–48, for the given equation:

(a) Write the right side of the equation in the form \( k(x + t)^2 + r \).

(b) Find the value of \( x \) where the right side of the equation attains its minimum value or its maximum value.

(c) Find the minimum or maximum value of the right side of the equation.

(d) Find the vertex of the graph of the equation.

45. \( y = x^2 + 7x + 12 \)

**SOLUTION**

(a) By completing the square, we can write

\[
\begin{align*}
x^2 + 7x + 12 &= (x + \frac{7}{2})^2 - \frac{49}{4} + 12 \\
&= (x + \frac{7}{2})^2 - \frac{1}{4}.
\end{align*}
\]

(b) The expression above shows that the value of the right side is minimized when \( x = -\frac{7}{2} \).

(c) The solution to part (a) shows that the minimum value of this expression is \(-\frac{1}{4}\), which occurs when \( x = -\frac{7}{2} \).

47. \( y = -2x^2 + 5x - 2 \)

**SOLUTION**

(a) By completing the square, we can write

\[
\begin{align*}
-2x^2 + 5x - 2 &= -2[x^2 - \frac{5}{2}x] - 2 \\
&= -2[(x - \frac{5}{4})^2 - \frac{25}{16}] - 2 \\
&= -2(x - \frac{5}{4})^2 + \frac{25}{8} - 2 \\
&= -2(x - \frac{5}{4})^2 + \frac{9}{8}.
\end{align*}
\]

(b) The expression above shows that the value of \(-2x^2 + 5x - 2\) is maximized when \( x = \frac{5}{4} \).

(c) The solution to part (a) shows that the maximum value of this expression is \(\frac{9}{8}\), which occurs when \( x = \frac{5}{4} \).

49. Find a constant \( c \) such that the graph of \( y = x^2 + 6x + c \) has its vertex on the \( x \)-axis.

**SOLUTION** First we find the vertex of the graph of \( y = x^2 + 6x + c \). To do this, complete the square:

\[
x^2 + 6x + c = (x + 3)^2 - 9 + c.
\]

Thus the value of \( x^2 + 6x + c \) is minimized when \( x = -3 \). When \( x = -3 \), the value of \( x^2 + 6x + c \)
equals \(-9+c\). Thus the vertex of \(y = x^2 + 6x + c\) is \((-3, -9+c)\).

The \(x\)-axis consists of the points whose second coordinate equals 0. Thus the vertex of the graph of \(y = x^2 + 6x + c\) will be on the \(x\)-axis when \(-9+c = 0\), or equivalently when \(c = 9\).

51. Find two numbers whose sum equals 10 and whose product equals 7.

**SOLUTION** Let’s call the two numbers \(s\) and \(t\). We want

\[s + t = 10 \quad \text{and} \quad st = 7.\]

Solving the first equation for \(s\), we have \(s = 10 - t\). Substituting this expression for \(s\) into the second equation gives \((10-t)t = 7\), which is equivalent to the equation

\[t^2 - 10t + 7 = 0.\]

Using the quadratic formula to solve this equation for \(t\) gives

\[t = \frac{-10 \pm \sqrt{100 - 4 \cdot 7}}{2} = \frac{10 \pm \sqrt{36}}{2} = 5 \pm 3\sqrt{2}.\]

Let’s choose the solution \(t = 5 + 3\sqrt{2}\). Plugging this value of \(t\) into the equation \(s = 10 - t\) then gives \(s = 5 - 3\sqrt{2}\).

Thus two numbers whose sum equals 10 and whose product equals 7 are \(5 - 3\sqrt{2}\) and \(5 + 3\sqrt{2}\).

**CHECK** To check that this solution is correct, note that

\[(5 - 3\sqrt{2}) + (5 + 3\sqrt{2}) = 10\]

and

\[(5 - 3\sqrt{2})(5 + 3\sqrt{2}) = 25 - 9 \cdot 2 = 7.\]

53. Find two positive numbers whose difference equals 3 and whose product equals 20.

**SOLUTION** Let’s call the two numbers \(s\) and \(t\). We want

\[s - t = 3 \quad \text{and} \quad st = 20.\]

Solving the first equation for \(s\), we have \(s = t + 3\). Substituting this expression for \(s\) into the second equation gives \((t + 3)t = 20\), which is equivalent to the equation

\[t^2 + 3t - 20 = 0.\]

Using the quadratic formula to solve this equation for \(t\) gives

\[t = \frac{-3 \pm \sqrt{3^2 + 4 \cdot 20}}{2} = \frac{-3 \pm \sqrt{89}}{2}.\]

Choosing the minus sign in the plus-or-minus expression above would lead to a negative value for \(t\). Because this exercise requires that \(t\) be positive, we choose \(t = \frac{-3 + \sqrt{89}}{2}\). Plugging this value of \(t\) into the equation \(s = t + 3\) then gives \(s = \frac{3 + \sqrt{89}}{2}\).

Thus two numbers whose difference equals 3 and whose product equals 20 are \(\frac{3 + \sqrt{89}}{2}\) and \(\frac{-3 + \sqrt{89}}{2}\).

55. Find the minimum value of \(x^2 - 6x + 2\).

**SOLUTION** By completing the square, we can write

\[x^2 - 6x + 2 = (x - 3)^2 - 9 + 2 = (x - 3)^2 - 7.\]

The expression above shows that the minimum value of \(x^2 - 6x + 2\) is \(-7\) (and that this minimum value occurs when \(x = 3\)).

57. Find the maximum value of \(7 - 2x - x^2\).

**SOLUTION** By completing the square, we can write

\[7 - 2x - x^2 = -(x^2 + 2x) + 7 = -(x + 1)^2 - 1 + 7 = -(x + 1)^2 + 8.\]

The expression above shows that the maximum value of \(7 - 2x - x^2\) is \(8\) (and that this maximum value occurs when \(x = -1\)).
LEARNING OBJECTIVES

By the end of this section you should be able to

- compute the areas of squares, rectangles, parallelograms, triangles, and trapezoids;
- explain how area changes when the coordinate axes are stretched;
- compute the area inside a circle;
- compute the area inside an ellipse.

You probably already have a good intuitive notion of area. In this section we will try to strengthen this intuition and build a good understanding of the formulas for the area of simple regions.

Squares, Rectangles, and Parallelograms

The most primitive notion of area is that a 1-by-1 square has area 1. If we can decompose a region into 1-by-1 squares, then the area of that region is the number of 1-by-1 squares into which it can be decomposed, as shown in the figure below:

If \( m \) is a positive integer, then an \( m \)-by-\( m \) square can be decomposed into \( m^2 \) squares of size 1-by-1. Thus it is no surprise that the area of an \( m \)-by-\( m \) square equals \( m^2 \).

The same formula holds for squares whose side length is not necessarily an integer, as shown below:

More generally, we have the following formula:

**Area of a square**

A square whose sides have length \( \ell \) has area \( \ell^2 \).

Consider a rectangle with base 3 and height 2, as shown here. This 3-by-2 rectangle can be decomposed into six 1-by-1 squares. Thus this rectangle has area 6.
Similarly, if \( b \) and \( h \) are positive integers, then a rectangle with base \( b \) and height \( h \) can be composed into \( bh \) squares of size 1-by-1, showing that the rectangle has area \( bh \). More generally, the same formula is valid even if the base and height are not integers.

**Area of a rectangle**

A rectangle with base \( b \) and height \( h \) has area \( bh \).

In the special case where the base equals the height, the formula for the area of a rectangle becomes the formula for the area of a square.

A parallelogram is a quadrilateral (a four-sided polygon) in which both pairs of opposite sides are parallel, as shown here.

To find the area of a parallelogram, select one of the sides and call its length the base. The opposite side of the parallelogram will have the same length. The height of the parallelogram is then defined to be the length of a line segment that connects these two sides and is perpendicular to both of them. Thus in the figure shown here, the parallelogram has base \( b \) and both vertical line segments have length equal to the height \( h \).

The two small triangles in the figure above have the same size and thus the same area. The rectangle in the figure above could be obtained from the parallelogram by moving the triangle on the right to the position of the triangle on the left. This shows that the parallelogram and the rectangle above have the same area. Because the area of the rectangle equals \( bh \), we thus have the following formula for the area of a parallelogram:

**Area of a parallelogram**

A parallelogram with base \( b \) and height \( h \) has area \( bh \).

**Triangles and Trapezoids**

To find the area of a triangle, select one of the sides and call its length the base. The height of the triangle is then defined to be the length of the perpendicular line segment that connects the opposite vertex to the side determining the base, as shown in the figure in the margin.

To derive the formula for the area of a triangle with base \( b \) and height \( h \), draw two line segments, each parallel to and the same length as one of the sides of the triangle, to form a parallelogram as in the figure below:

The triangle has been extended to a parallelogram by adjoining a second triangle with the same side lengths as the original triangle.

Use the same units for the base and the height. The unit of measurement for area is then the square of the unit used for these lengths. For example, a rectangle with base 3 feet and height 2 feet has area 6 square feet.
The parallelogram above has base $b$ and height $h$ and hence has area $bh$. The original triangle has area equal to half the area of the parallelogram. Thus we obtain the following formula:

**Area of a triangle**

A triangle with base $b$ and height $h$ has area $\frac{1}{2}bh$.

---

**EXAMPLE 1**

Find the area of the triangle whose vertices are $(1,0)$, $(9,0)$, and $(7,3)$.

**SOLUTION** Choose the side connecting $(1,0)$ and $(9,0)$ as the base of this triangle. Thus this triangle has base $9 - 1$, which equals 8.

The height of this triangle is the length of the red line shown here; this height equals the second coordinate of the vertex $(7,3)$. In other words, this triangle has height 3.

Thus this triangle has area $\frac{1}{2} \cdot 8 \cdot 3$, which equals 12.

Consider the special case where our triangle happens to be a right triangle, with the right angle between sides of length $a$ and $b$. Choosing $b$ to be the base of the triangle, we see that the height of this triangle equals $a$. Thus in this case the area of the triangle equals $\frac{1}{2}ab$.

A **trapezoid** is a quadrilateral that has at least one pair of parallel sides, as for example shown here. The lengths of a pair of opposite parallel sides are called the **bases**, which are denoted below by $b_1$ and $b_2$. The **height** of the trapezoid, denoted $h$ below, is then defined to be the length of a line segment that connects these two sides and that is perpendicular to both of them.

The diagonal in the figure here divides the trapezoid into two triangles. The lower triangle has base $b_1$ and height $h$; thus the lower triangle has area $\frac{1}{2}b_1h$. The upper triangle has base $b_2$ and height $h$; thus the upper triangle has area $\frac{1}{2}b_2h$. The area of the trapezoid is the sum of the areas of these two triangles. Thus the area of the trapezoid equals $\frac{1}{2}b_1h + \frac{1}{2}b_2h$. Factoring out the $\frac{1}{2}$ and the $h$ in this expression gives the following formula:

**Area of a trapezoid**

A trapezoid with bases $b_1$, $b_2$ and height $h$ has area $\frac{1}{2}(b_1 + b_2)h$.

Note that $\frac{1}{2}(b_1 + b_2)$ is just the average of the two bases of the trapezoid. In the special case where the trapezoid is a parallelogram, the two bases are equal and we are back to the familiar formula that the area of a parallelogram equals the base times the height.
Find the area of the region in the \(xy\)-plane under the line \(y = 2x\), above the \(x\)-axis, and between the lines \(x = 2\) and \(x = 5\).

**EXAMPLE 2**

**SOLUTION**

The line \(x = 2\) intersects the line \(y = 2x\) at the point (2, 4). The line \(x = 5\) intersects the line \(y = 2x\) at the point (5, 10).

Thus the region in question is the trapezoid shown above. The parallel sides of this trapezoid (the two vertical sides) have lengths 4 and 10, and thus this trapezoid has bases 4 and 10. As can be seen from the figure above, this trapezoid has height 3 (note that in this trapezoid, the height is the length of the horizontal side). Thus the area of this trapezoid is \(\frac{1}{2} \cdot (4 + 10) \cdot 3\), which equals 21.

**Stretching**

Suppose a square whose sides have length 1 has its sides tripled in length, resulting in a square whose sides have length 3, as shown here. You can think of this transformation as stretching both vertically and horizontally by a factor of 3. This transformation increases the area of the square by a factor of 9.

Consider now the transformation that stretches horizontally by a factor of 3 and stretches vertically by a factor of 2. This transformation changes a square whose sides have length 1 into a rectangle with base 3 and height 2, as shown here. Thus the area has been increased by a factor of 6.

More generally, suppose \(c, d\) are positive numbers, and consider the transformation that stretches horizontally by a factor of \(c\) and stretches vertically by a factor of \(d\). This transformation changes a square whose sides have length 1 into a rectangle with base \(c\) and height \(d\), as shown here. Thus the area has been increased by a factor of \(cd\).

We need not restrict our attention to squares. The transformation that stretches horizontally by a factor of \(c\) and stretches vertically by a factor of \(d\) will change any region into a new region whose area has been changed by a factor of \(cd\). This result follows from the result for squares, because any region can be approximated by a union of squares, as shown here for a triangle. Here is the formal statement of this result:

**Area Stretch Theorem**

Suppose \(R\) is a region in the coordinate plane and \(c, d\) are positive numbers. Let \(R'\) be the region obtained from \(R\) by stretching horizontally by a factor of \(c\) and stretching vertically by a factor of \(d\). Then

the area of \(R'\) equals \(cd\) times the area of \(R\).
Circles and Ellipses

Consider the region inside a circle of radius 1 centered at the origin. If we stretch both horizontally and vertically by a factor of \( r \), this region becomes the region inside the circle of radius \( r \) centered at the origin, as shown in the figure below for \( r = 2 \).

Stretching both horizontally and vertically by a factor of 2 transforms a circle of radius 1 into a circle of radius 2.

Let \( p \) denote the area inside a circle of radius 1. The Area Stretch Theorem implies that the area inside a circle of radius \( r \) equals \( r^2 p \), which we write in the more familiar form \( pr^2 \). We need to find the value of \( p \).

To find \( p \), consider a circle of radius 1 surrounded by a slightly larger circle with radius \( r \), as shown in the margin. Cut out the region between the two circles, then cut a slit in it and unwind it into the shape of a trapezoid (this requires a tiny bit of distortion) as shown below.

The upper base of the trapezoid is the circumference of the circle of radius 1; the bottom base is the circumference of the circle of radius \( r \).

The trapezoid has height \( r - 1 \), which is the distance between the two original circles. The trapezoid has bases \( 2\pi r \) and \( 2\pi \), corresponding to the circumferences of the two circles. Thus the trapezoid has area

\[
\frac{1}{2}(2\pi r + 2\pi)(r - 1),
\]

which equals \( \pi (r + 1)(r - 1) \), which equals \( \pi (r^2 - 1) \).

The area inside the larger circle equals the area inside the circle of radius 1 plus the area of the region between the two circles. In other words, the area inside the larger circle equals \( p + \pi (r^2 - 1) \). The area inside the larger circle also equals \( pr^2 \), because the larger circle has radius \( r \). Thus we have

\[
pr^2 = p + \pi (r^2 - 1).
\]

Subtracting \( p \) from both sides, we get

\[
p(r^2 - 1) = \pi (r^2 - 1).
\]

Thus \( p = \pi \). In other words, the area inside a circle of radius \( r \) equals \( \pi r^2 \).
We have derived the following formula:

**Area inside a circle**

The area inside a circle of radius $r$ is $\pi r^2$.

Thus to find the area inside a circle, we must first find the radius of the circle. Finding the radius sometimes requires a preliminary algebraic manipulation such as completing the square, as shown in the following example.

Consider the circle described by the equation

$$x^2 - 8x + y^2 + 6y = 4.$$ 

(a) Find the center of this circle.
(b) Find the radius of this circle.
(c) Find the circumference of this circle.
(d) Find the area inside this circle.

**SOLUTION**

To obtain the desired information about the circle, we put its equation in a standard form. This can be done by completing the square:

$$4 = x^2 - 8x + y^2 + 6y
= (x - 4)^2 - 16 + (y + 3)^2 - 9
= (x - 4)^2 + (y + 3)^2 - 25.$$ 

Adding 25 to the first and last sides above shows that the circle is described by the equation

$$(x - 4)^2 + (y + 3)^2 = 29.$$ 

(a) The equation above shows that the center of the circle is $(4, -3)$.
(b) The equation above shows that the radius of the circle is $\sqrt{29}$.
(c) Because the circle has radius $\sqrt{29}$, its circumference is $2\sqrt{29}\pi$.
(d) Because the circle has radius $\sqrt{29}$, its area is $29\pi$.

In Section 2.3 we saw that the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

is obtained from the circle of radius 1 centered at the origin by stretching horizontally by a factor of 5 and stretching vertically by factor of 3.
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Stretching horizontally by a factor of 5 and vertically by a factor of 3 transforms the circle on the left into the ellipse on the right.

Because $5 \cdot 3 = 15$, the Area Stretch Theorem tells us that the area inside this ellipse equals 15 times the area inside the circle of radius 1. Because the area inside a circle of radius 1 is $\pi$, we conclude that the area inside this ellipse is $15\pi$.

More generally, suppose $a$ and $b$ are positive numbers. Suppose the circle of radius 1 centered at the origin is stretched horizontally by a factor of $a$ and stretched vertically by a factor of $b$. As we saw in Section 2.3, the equation of the resulting ellipse in the $xy$-plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The Area Stretch Theorem now gives us the following formula:

**Area inside an ellipse**

Suppose $a$ and $b$ are positive numbers. Then the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $\pi ab$.

**Example 4** Find the area inside the ellipse

$$4x^2 + 5y^2 = 3.$$

**Solution** To put the equation of this ellipse in the form given by the area formula, begin by dividing both sides by 3, and then force the equation into the desired form, as follows:
1 = \frac{4}{3}x^2 + \frac{2}{3}y^2
\begin{align*}
= \frac{x^2}{4} + \frac{y^2}{3} \\
= \frac{x^2}{(\sqrt{3})^2} + \frac{y^2}{\sqrt{3}^2}.
\end{align*}
Thus the area inside the ellipse is \( \pi \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{5}{3}} \), which equals \( \frac{3\sqrt{5}}{10} \pi \).

Ellipses need not be centered at the origin. For example, the equation
\[ (x-5)^2 \left(\frac{9}{a^2}\right) + (y-7)^2 \left(\frac{16}{b^2}\right) = 1 \]
represents an ellipse centered at a point \((5, 7)\). This ellipse is obtained by shifting the ellipse whose equation is \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \) right 5 units and up 7 units. The formula above tells us that the area inside the ellipse
\[ \frac{(x-5)^2}{9} + \frac{(y-7)^2}{16} = 1 \]
is 12\(\pi\), and thus the area inside the ellipse
\[ \frac{(x-5)^2}{9} + \frac{(y-7)^2}{16} = 1 \]
is also 12\(\pi\).

More generally, if \(a\) and \(b\) are positive numbers, then the equation
\[ \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \]
represents an ellipse centered at a point \((h, k)\). This ellipse is obtained by shifting the ellipse whose equation is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Thus the area inside the ellipse
\[ \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \]
is \(\pi ab\).

**EXERCISES**

1. Find the area of a triangle that has two sides of length 6 and one side of length 10.
2. Find the area of a triangle that has two sides of length 6 and one side of length 4.
3. (a) Find the distance from the point \((2, 3)\) to the line containing the points \((-2, -1)\) and \((5, 4)\).
   (b) Use the information from part (a) to find the area of the triangle whose vertices are \((2, 3), (-2, -1), \) and \((5, 4)\).
4. (a) Find the distance from the point \((3, 4)\) to the line containing the points \((1, 5)\) and \((-2, 2)\).
   (b) Use the information from part (a) to find the area of the triangle whose vertices are \((3, 4), (1, 5), \) and \((-2, 2)\).
5. Find the area of the triangle whose vertices are \((2, 0), (9, 0), \) and \((4, 5)\).
6. Find the area of the triangle whose vertices are \((-3, 0), (2, 0), \) and \((4, 3)\).
7. Suppose \((2, 3), (1, 1), \) and \((7, 1)\) are three vertices of a parallelogram, two of whose sides are shown here.
   (a) Find the fourth vertex of this parallelogram.
   (b) Find the area of this parallelogram.
8. Suppose \((3, 4), (2, 1),\) and \((6, 1)\) are three vertices of a parallelogram, two of whose sides are shown here. 

(a) Find the fourth vertex of this parallelogram. 

(b) Find the area of this parallelogram. 

9. Find the area of this trapezoid, whose vertices are \((1, 1), (7, 1), (5, 3),\) and \((2, 3)\). 

10. Find the area of this trapezoid, whose vertices are \((2, 1), (6, 1), (8, 4),\) and \((1, 4)\). 

11. Find the area of the region in the \(xy\)-plane under the line \(y = \frac{x}{2},\) above the \(x\)-axis, and between the lines \(x = 2\) and \(x = 6\). 

12. Find the area of the region in the \(xy\)-plane under the line \(y = 3x + 1,\) above the \(x\)-axis, and between the lines \(x = 1\) and \(x = 5\). 

13. Let \(f(x) = |x|\). Find the area of the region in the \(xy\)-plane under the graph of \(f,\) above the \(x\)-axis, and between the lines \(x = -2\) and \(x = 5\). 

14. Let \(f(x) = |2x|\). Find the area of the region in the \(xy\)-plane under the graph of \(f,\) above the \(x\)-axis, and between the lines \(x = -3\) and \(x = 4\). 

15. Find the area inside a circle with diameter 7. 

16. Find the area inside a circle with diameter 9. 

17. Find the area inside a circle with circumference 5 feet. 

18. Find the area inside a circle with circumference 7 yards. 

19. Find the area inside the circle whose equation is \(x^2 - 6x + y^2 + 10y = 1.\) 

20. Find the area inside the circle whose equation is \(x^2 + 5x + y^2 - 3y = 1.\) 

21. Find a number \(t\) such that the area inside the circle \(3x^2 + 3y^2 = t\) is 8. 

22. Find a number \(t\) such that the area inside the circle \(5x^2 + 5y^2 = t\) is 2. 

23. What is the area of a DVD disk, not counting the hole? 

24. What is the area of a DVD disk, not counting the hole and the unusable circular ring with width 1.5 cm that surrounds the hole? 

25. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 3100 megabytes of data storage? 

26. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 4200 megabytes of data storage? 

27. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 200 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?
28. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 350 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?

29. Find the area of the region in the \(xy\)-plane under the curve \(y = \sqrt{4 - x^2}\) (with \(-2 \leq x \leq 2\)) and above the \(x\)-axis.

30. Find the area of the region in the \(xy\)-plane under the curve \(y = \sqrt{9 - x^2}\) (with \(-3 \leq x \leq 3\)) and above the \(x\)-axis.

31. Using the answer from Exercise 29, find the area of the region in the \(xy\)-plane under the curve \(y = 3\sqrt{4 - x^2}\) (with \(-2 \leq x \leq 2\)) and above the \(x\)-axis.

32. Using the answer from Exercise 30, find the area of the region in the \(xy\)-plane under the curve \(y = 5\sqrt{9 - x^2}\) (with \(-3 \leq x \leq 3\)) and above the \(x\)-axis.

33. Using the answer from Exercise 29, find the area of the region in the \(xy\)-plane under the curve \(y = \sqrt{4 - x^2}\) (with \(-6 \leq x \leq 6\)) and above the \(x\)-axis.

34. Using the answer from Exercise 30, find the area of the region in the \(xy\)-plane under the curve \(y = \sqrt{9 - x^2}\) (with \(-12 \leq x \leq 12\)) and above the \(x\)-axis.

35. Find the area of the region in the \(xy\)-plane under the curve
\[
y = 1 + \sqrt{4 - x^2},
\]
above the \(x\)-axis, and between the lines \(x = -2\) and \(x = 2\).

36. Find the area of the region in the \(xy\)-plane under the curve
\[
y = 2 + \sqrt{9 - x^2},
\]
above the \(x\)-axis, and between the lines \(x = -3\) and \(x = 3\).

In Exercises 37–44, find the area inside the ellipse in the \(xy\)-plane determined by the given equation.

37. \[
\frac{x^2}{7} + \frac{y^2}{16} = 1
\]
38. \[
\frac{x^2}{9} + \frac{y^2}{5} = 1
\]
39. \[
2x^2 + 3y^2 = 1
\]
40. \[
10x^2 + 7y^2 = 1
\]
41. \[
3x^2 + 2y^2 = 7
\]
42. \[
5x^2 + 9y^2 = 3
\]
43. \[
3x^2 + 4x + 2y^2 + 3y = 2
\]
44. \[
4x^2 + 2x + 5y^2 + y = 2
\]
45. Find a positive number \(c\) such that the area inside the ellipse
\[
2x^2 + cy^2 = 5
\]
is 3.

46. Find a positive number \(c\) such that the area inside the ellipse
\[
cx^2 + 7y^2 = 3
\]
is 2.

47. Find numbers \(a\) and \(b\) such that \(a > b\), \(a + b = 15\), and the area inside the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
is \(36\pi\).

48. Find numbers \(a\) and \(b\) such that \(a > b\), \(a + b = 5\), and the area inside the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
is \(3\pi\).

49. Find a number \(t\) such that the area inside the ellipse
\[
4x^2 + 9y^2 = t
\]
is 5.

50. Find a number \(t\) such that the area inside the ellipse
\[
2x^2 + 3y^2 = t
\]
is 7.
PROBLEMS

51. Explain why a square yard contains 9 square feet.

52. Explain why a square foot contains 144 square inches.

53. Find a formula that gives the area of a square in terms of the length of the diagonal of the square.

54. Find a formula that gives the area of a square in terms of the perimeter.

55. Suppose $a$ and $b$ are positive numbers. Draw a figure of a square whose sides have length $a + b$. Partition this square into a square whose sides have length $a$, a square whose sides have length $b$, and two rectangles in a way that illustrates the identity

$$(a + b)^2 = a^2 + 2ab + b^2.$$

56. Find an example of a parallelogram whose area equals 10 and whose perimeter equals 16 (give the coordinates for all four vertices of your parallelogram).

57. Show that an equilateral triangle with sides of length $r$ has area $\frac{\sqrt{3}}{4} r^2$.

58. Show that an equilateral triangle with area $A$ has sides of length $\frac{2\sqrt{3}A}{3}$.

59. Suppose $0 < a < b$. Show that the area of the region under the line $y = x$, above the $x$-axis, and between the lines $x = a$ and $x = b$ is $\frac{b^2 - a^2}{2}$.

60. Show that the area inside a circle with circumference $c$ is $\frac{c^2}{4\pi}$.

61. Find a formula that gives the area inside a circle in terms of the diameter of the circle.

62. In ancient China and Babylonia, the area inside a circle was said to be one-half the radius times the circumference. Show that this formula agrees with our formula for the area inside a circle.

63. Suppose $a$, $b$, and $c$ are positive numbers. Show that the area inside the ellipse

$$ax^2 + by^2 = c$$

is $\pi \frac{c}{4ab}$.

64. Consider the following figure, which is drawn accurately to scale:

(a) Show that the right triangle whose vertices are $(0,0)$, $(20,0)$, and $(20,9)$ has area 90.

(b) Show that the yellow right triangle has area 27.5.

(c) Show that the blue rectangle has area 45.

(d) Show that the red right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect that parts (a) and (e) will have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.

65. The figure below illustrates a circle of radius 1 enclosed within a square. By comparing areas, explain why this figure shows that $\pi < 4$. 
WORKED-OUT SOLUTIONS to Odd-numbered Exercises

1. Find the area of a triangle that has two sides of length 6 and one side of length 10.

**SOLUTION** By the Pythagorean Theorem (see figure below), the height of this triangle equals \( \sqrt{6^2 - 5^2} \), which equals \( \sqrt{11} \).

![A triangle that has two sides of length 6 and one side of length 10.](image)

Thus the area of this triangle equals \( 5 \sqrt{11} \).

2. (a) Find the area of a triangle that has two sides of length 6 and one side of length 10.

To find where this line intersects the line containing the points \((-2, -1)\) and \((5, 4)\), we need to solve the equation

\[
\frac{y - 3}{x - 2} = \frac{-7}{5},
\]

which can be rewritten as

\[
y = -\frac{7}{5}x + \frac{29}{5}.
\]

To find where this line intersects the line containing the points \((-2, -1)\) and \((5, 4)\), we need to solve the equation

\[
\frac{5}{7}x + \frac{3}{7} = -\frac{7}{5}x + \frac{29}{5}.
\]

Simple algebra shows that the solution to this equation is \( x = \frac{94}{37} \). Plugging this value of \( x \) into the equation of either line shows that \( y = \frac{83}{37} \). Thus the two lines intersect at the point \((\frac{94}{37}, \frac{83}{37})\).

Thus the distance from the point \((2, 3)\) to the line containing the points \((-2, -1)\) and \((5, 4)\) is the distance from the point \((2, 3)\) to the point \((\frac{94}{37}, \frac{83}{37})\). This distance equals

\[
\sqrt{(2 - \frac{94}{37})^2 + (3 - \frac{83}{37})^2},
\]

which equals \( \frac{\sqrt{4}}{37} \), which equals \( \sqrt{\frac{2}{37}} \).

(b) We will consider the line segment connecting the points \((-2, -1)\) and \((5, 4)\) to be the base of this triangle. In part (a), we found that the height of this triangle equals \( 4 \sqrt{\frac{2}{37}} \).

![The triangle with vertices (2,3), (-2,-1), and (5,4), with a line segment showing its height.](image)

The base of the triangle is the distance between the points \((-2, -1)\) and \((5, 4)\). This distance equals \( \sqrt{14} \). Thus the area of the triangle (one-half the base times the height) equals

\[
\frac{1}{2} \cdot \sqrt{14} \cdot (4 \sqrt{\frac{2}{37}}),
\]
which equals 4.

[There are easier ways to find the area of this triangle, but the technique used here gives you practice with several important concepts.]

5. Find the area of the triangle whose vertices are (2, 0), (9, 0), and (4, 5).

**SOLUTION** Choose the side connecting (2, 0) and (9, 0) as the base of this triangle. Thus the triangle below has base 9 − 2, which equals 7.

![Diagram of triangle](image)

The height of this triangle is the length of the red line shown here; this height equals the second coordinate of the vertex (4, 5). In other words, this triangle has height 5.

Thus this triangle has area \( \frac{1}{2} \cdot 7 \cdot 5 \), which equals \( \frac{35}{2} \).

7. Suppose (2, 3), (1, 1), and (7, 1) are three vertices of a parallelogram, two of whose sides are shown here.

(a) Find the fourth vertex of this parallelogram.

(b) Find the area of this parallelogram.

**SOLUTION**

(a) Consider the horizontal side of the parallelogram connecting the points (1, 1) and (7, 1). This side has length 6. Thus the opposite side, which connects the point (2, 3) and the fourth vertex, must also be horizontal and have length 6. Thus the second coordinate of the fourth vertex is the same as the second coordinate of (2, 3), and the first coordinate of the fourth vertex is obtained by adding 6 to the first coordinate of (2, 3). Hence the fourth vertex equals (8, 3).

(b) The base of this parallelogram is the length of the side connecting the points (1, 1) and (7, 1), which equals 6. The height of this parallelogram is the length of a vertical line segment connecting the two horizontal sides. Because one of the horizontal sides lies on the line \( y = 1 \) and the other horizontal side lies on the line \( y = 3 \), a vertical line segment connecting these two sides will have length 2. Thus the parallelogram has height 2. Because this parallelogram has base 6 and height 2, it has area 12.

9. Find the area of this trapezoid, whose vertices are (1, 1), (7, 1), (5, 3), and (2, 3).

**SOLUTION** One base of this trapezoid is the length of the side connecting the points (1, 1) and (7, 1), which equals 6. The other base of this trapezoid is the length of the side connecting the points (5, 3) and (2, 3), which equals 3.

The height of this trapezoid is the length of a vertical line segment connecting the two horizontal sides. Because one of the horizontal sides lies on the line \( y = 1 \) and the other horizontal side lies on the line \( y = 3 \), a vertical line segment connecting these two sides will have length 2. Thus the trapezoid has height 2.

Because this trapezoid has bases 6 and 3 and has height 2, it has area \( \frac{1}{2} (6 + 3) \cdot 2 \), which equals 9.

11. Find the area of the region in the \( xy \)-plane under the line \( y = \frac{x}{2} \), above the \( x \)-axis, and between the lines \( x = 2 \) and \( x = 6 \).

**SOLUTION** The line \( x = 2 \) intersects the line \( y = \frac{x}{2} \) at the point (2, 1). The line \( x = 6 \) intersects the line \( y = \frac{x}{2} \) at the point (6, 3).
Thus the region in question is the trapezoid shown above. The parallel sides of this trapezoid (the two vertical sides) have lengths 1 and 3, and thus this trapezoid has bases 1 and 3. As can be seen from the figure above, this trapezoid has height 4. Thus the area of this trapezoid is \[ \frac{1}{2} \cdot (1 + 3) \cdot 4, \] which equals 8.

13. Let \( f(x) = |x| \). Find the area of the region in the \( xy \)-plane under the graph of \( f \), above the \( x \)-axis, and between the lines \( x = -2 \) and \( x = 5 \).

**SOLUTION**
The region under consideration is the union of two triangles, as shown here.

One of the triangles has base 2 and height 2 and thus has area 2. The other triangle has base 5 and height 5 and thus has area \( \frac{25}{2} \). Thus the area of the region under consideration equals \( 2 + \frac{25}{2} \), which equals \( \frac{29}{2} \).

15. Find the area inside a circle with diameter 7.

**SOLUTION** A circle with diameter 7 has radius \( \frac{7}{2} \). Thus the area inside this circle is \( \pi \left( \frac{7}{2} \right)^2 \), which equals \( \frac{49\pi}{4} \).

17. Find the area inside a circle with circumference 5 feet.

**SOLUTION** Let \( r \) denote the radius of this circle in feet. Then \( 2\pi r = 5 \), which implies that \( r = \frac{5}{2\pi} \). Thus the area inside this circle is \( \pi \left( \frac{5}{2\pi} \right)^2 \) square feet, which equals \( \frac{25}{4\pi} \) square feet.

19. Find the area inside the circle whose equation is

\[ x^2 - 6x + y^2 + 10y = 1. \]

**SOLUTION** To find the radius of the circle given by the equation above, we complete the square, as follows:

\[ \begin{align*}
1 &= x^2 - 6x + y^2 + 10y \\
&= (x - 3)^2 - 9 + (y + 5)^2 - 25 \\
&= (x - 3)^2 + (y + 5)^2 - 34.
\end{align*} \]

Adding 34 to both sides of this equation gives

\[ (x - 3)^2 + (y + 5)^2 = 35. \]

Thus we see that this circle is centered at \( (3, -5) \) (which is irrelevant for this exercise) and that it has radius \( \sqrt{35} \). Thus the area inside this circle equals \( \pi \sqrt{35} \), which equals \( 35\pi \).

21. Find a number \( t \) such that the area inside the circle

\[ 3x^2 + 3y^2 = t \]

is 8.

**SOLUTION** Rewriting the equation above as

\[ x^2 + y^2 = \left( \frac{\sqrt{t}}{3} \right)^2, \]

we see that this circle has radius \( \frac{\sqrt{t}}{3} \). Thus the area inside this circle is \( \pi \left( \frac{\sqrt{t}}{3} \right)^2 \), which equals \( \frac{\pi t}{3} \). We want this area to equal 8, which means we need to solve the equation \( \frac{\pi t}{3} = 8 \). Thus

\[ t = \frac{24}{\pi}. \]

**Use the following information for Exercises 23–28:**

23. What is the area of a DVD disk, not counting the hole?

**SOLUTION** The DVD disk has radius 6 cm (because the diameter is 12 cm). The area inside a circle with radius 6 cm is \( 36\pi \) square cm.

The area of the hole is \( 0.75^2\pi \) square cm, which is \( 0.5625\pi \) square cm.

Subtracting \( 0.5625\pi \) from \( 36\pi \), we see that the DVD disk has area \( 34.4375\pi \) square cm, which is approximately 111.33 square cm.

25. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 3100 megabytes of data storage?
Because 50.2 megabytes of data can be stored in each square cm of usable surface, the usable surface must have area at least \( \frac{3100}{50.2} \) square cm, or about 61.753 square cm.

If the outer radius of the circular ring of usable area is \( r \), then the usable area will be \( \pi r^2 - 2.25 \pi \). Thus we solve the equation

\[
\pi r^2 - 2.25 \pi = 61.753
\]

for \( r \), getting \( r \approx 4.97 \text{ cm} \).

27. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 200 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?

Because 50.2 megabytes of data can be stored in each square cm of usable surface, the usable surface must have area at least \( \frac{200}{50.2} \) square cm, or about 3.984 square cm.

If the inner radius of the circular ring of usable area is \( r \), then the usable area will be \( \frac{5}{9} \pi r^2 \). Thus we solve the equation

\[
\frac{5}{9} \pi r^2 = 3.984
\]

for \( r \), getting \( r \approx 5.79 \text{ cm} \).

29. Find the area of the region in the \( xy \)-plane under the curve \( y = \sqrt{4 - x^2} \) (with \(-2 \leq x \leq 2\)) and above the \( x \)-axis.

SOLUTION

Square both sides of the equation

\[
y = \sqrt{4 - x^2}
\]

and then add \( x^2 \) to both sides.

This gives the equation \( x^2 + y^2 = 4 \), which is the equation of a circle of radius 2 centered at the origin. However, the equation \( y = \sqrt{4 - x^2} \) forces \( y \) to be nonnegative, and thus we have only the top half of the circle. Thus the region in question, which is shown above, has half the area inside a circle of radius 2. Hence the area of this region is \( \frac{1}{2} \pi \cdot 2^2 \), which equals \( 2\pi \).

31. Using the answer from Exercise 29, find the area of the region in the \( xy \)-plane under the curve \( y = 3\sqrt{4 - x^2} \) (with \(-2 \leq x \leq 2\)) and above the \( x \)-axis.

SOLUTION

The region in this exercise is obtained from the region in Exercise 29 by stretching vertically by a factor of 3. Thus by the Area Stretch Theorem, the area of this region is \( 3 \times \text{area of the region in Exercise 29} \). Thus this region has area \( 6\pi \).

33. Using the answer from Exercise 29, find the area of the region in the \( xy \)-plane under the curve \( y = \sqrt{4 - \frac{x^2}{9}} \) (with \(-6 \leq x \leq 6\)) and above the \( x \)-axis.

SOLUTION

Define a function \( f \) with domain the interval \([-2, 2]\) by \( f(x) = \sqrt{4 - x^2} \). Define a function \( h \) with domain the interval \([-6, 6]\) by \( h(x) = f\left(\frac{x}{3}\right) \). Thus

\[
h(x) = f\left(\frac{x}{3}\right) = \sqrt{4 - \left(\frac{x}{3}\right)^2} = \sqrt{4 - \frac{x^2}{9}}.
\]

Hence the graph of \( h \) is obtained by horizontally stretching the graph of \( f \) by a factor of 3 (see Section 3.2). Thus the region in this exercise is obtained from the region in Exercise 29 by stretching horizontally by a factor of 3.
Thus by the Area Stretch Theorem, this region has area $6\pi$.

35. Find the area of the region in the $xy$-plane under the curve

$$y = 1 + \sqrt{4 - x^2},$$

above the $x$-axis, and between the lines $x = -2$ and $x = 2$.

**SOLUTION** The curve $y = 1 + \sqrt{4 - x^2}$ is obtained by shifting the curve $y = \sqrt{4 - x^2}$ up 1 unit.

Thus we have the region above, which should be compared to the region shown in the solution to Exercise 29.

To find the area of this region, we break it into two parts. One part consists of the rectangle shown above that has base 4 and height 1 (and thus has area 4); the other part is obtained by shifting the region in Exercise 29 up 1 unit (and thus has area $2\pi$, which is the area of the region in Exercise 29). Adding together the areas of these two parts, we conclude that the region shown above has area $4 + 2\pi$.

**In Exercises 37–44, find the area inside the ellipse in the $xy$-plane determined by the given equation.**

37. $\frac{x^2}{7} + \frac{y^2}{16} = 1$

**SOLUTION** Rewrite the equation of this ellipse as

$$\frac{x^2}{\sqrt{7}} + \frac{y^2}{4} = 1.$$

Thus the area inside this ellipse is $4\sqrt{7}\pi$.

39. $2x^2 + 3y^2 = 1$

**SOLUTION** Rewrite the equation of this ellipse in the form given by the area formula, as follows:

$$1 = 2x^2 + 3y^2$$

$$= \frac{x^2}{\frac{1}{2}} + \frac{y^2}{\frac{1}{3}}$$

$$= \frac{x^2}{\sqrt{\frac{2}{7}}} + \frac{y^2}{\sqrt{\frac{3}{7}}}.$$

Thus the area inside the ellipse is $\pi \cdot \sqrt{\frac{2}{7}} \cdot \sqrt{\frac{3}{7}}$, which equals $\frac{7\pi}{\sqrt{6}}$. Multiplying numerator and denominator by $\sqrt{6}$, we see that we could also express this area as $\frac{7\sqrt{6}\pi}{6}$.

41. $3x^2 + 2y^2 = 7$

**SOLUTION** To put the equation of the ellipse in the form given by the area formula, begin by dividing both sides by 7, and then force the equation into the desired form, as follows:

$$1 = \frac{3}{7}x^2 + \frac{2}{7}y^2$$

$$= \frac{x^2}{\frac{7}{3}} + \frac{y^2}{\frac{7}{2}}$$

Thus the area inside the ellipse is $\pi \cdot \sqrt{\frac{7}{3}} \cdot \sqrt{\frac{7}{2}}$, which equals $\frac{7\pi}{\sqrt{6}}$. Multiplying numerator and denominator by $\sqrt{6}$, we see that we could also express this area as $\frac{7\sqrt{6}\pi}{6}$.

43. $3x^2 + 4x + 2y^2 + 3y = 2$

**SOLUTION** To put the equation of this ellipse in a standard form, we complete the square, as follows:

$$2 = 3x^2 + 4x + 2y^2 + 3y$$

$$= 3[x^2 + \frac{4}{3}x] + 2[y^2 + \frac{3}{2}y]$$

$$= 3[(x + \frac{2}{3})^2 - \frac{4}{9}] + 2[(y + \frac{3}{4})^2 - \frac{9}{16}]$$

$$= 3(x + \frac{2}{3})^2 - \frac{4}{3} + 2(y + \frac{3}{4})^2 - \frac{9}{8}$$

$$= 3(x + \frac{2}{3})^2 + 2(y + \frac{3}{4})^2 - \frac{59}{24}.$$  

Adding $\frac{69}{24}$ to both sides of this equation gives

$$3(x + \frac{2}{3})^2 + 2(y + \frac{3}{4})^2 = \frac{107}{24}.$$  

Now multiplying both sides of this equation by $\frac{24}{107}$ gives
\[
\frac{72}{107} \left( x + \frac{2}{3} \right)^2 + \frac{48}{107} \left( y + \frac{3}{4} \right)^2 = 1.
\]

We rewrite this equation in the form
\[
\left( \frac{x + \frac{2}{3}}{\frac{\sqrt{72}}{107}} \right)^2 + \left( \frac{y + \frac{3}{4}}{\frac{\sqrt{48}}{107}} \right)^2 = 1.
\]

Thus the area inside this ellipse is
\[
\pi \frac{\sqrt{107} \cdot \sqrt{72}}{\sqrt{107} \cdot \sqrt{48}}.
\]

Because \( \sqrt{72} \sqrt{48} = \sqrt{36 \cdot 2 \cdot 16 \cdot 3} = 6 \sqrt{2} \cdot 4 \sqrt{3} \), this equals
\[
\pi \frac{107}{24 \sqrt{6}}.
\]

Multiplying numerator and denominator by \( \sqrt{6} \) also allows us to express this area as
\[
\pi \frac{107 \sqrt{6}}{144}.
\]

47. Find numbers \( a \) and \( b \) such that \( a > b \), \( a + b = 15 \), and the area inside the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
is \( 36 \pi \).

**SOLUTION** The area inside the ellipse is \( \pi ab \). Thus we need to solve the simultaneous equations
\[
a + b = 15 \quad \text{and} \quad ab = 36.
\]
The first equation can be rewritten as \( b = 15 - a \), and this value for \( b \) can then be substituted into the second equation, giving the equation
\[
a(15 - a) = 36.
\]

This is equivalent to the equation \( a^2 - 15a + 36 = 0 \), whose solutions (which can be found through either factoring or the quadratic formula) are \( a = 3 \) and \( a = 12 \). Choosing \( a = 3 \) gives \( b = 15 - a = 12 \), which violates the condition that \( a > b \). Choosing \( a = 12 \) gives \( b = 15 - 12 = 3 \). Thus the only solution to this exercise is \( a = 12, b = 3 \).

48. Find a number \( t \) such that the area inside the ellipse
\[
4x^2 + 9y^2 = t
\]
is \( 5 \).

**SOLUTION** Dividing the equation above by \( t \), we have
\[
1 = \frac{4}{t} x^2 + \frac{9}{t} y^2
\]
\[
= \frac{x^2}{\frac{t}{4}} + \frac{y^2}{\frac{t}{9}}
\]
\[
= \frac{x^2}{\left( \frac{\sqrt{t}}{2} \right)^2} + \frac{y^2}{\left( \frac{\sqrt{t}}{3} \right)^2}.
\]

Thus the area inside this ellipse is
\[
\pi \frac{\sqrt{t}}{2} \cdot \frac{\sqrt{t}}{3},
\]

which equals \( \frac{\pi t}{6} \). Hence we want \( \frac{\pi t}{6} = 5 \), which means that \( t = \frac{30}{\pi} \).
CHAPTER SUMMARY

To check that you have mastered the most important concepts and skills covered in this chapter, make sure that you can do each item in the following list:

- Locate points on the coordinate plane.
- Compute the distance between two points.
- Find the equation of a line given its slope and a point on it.
- Find the equation of a line given two points on it.
- Find the equation of a line parallel to a given line and containing a given point.
- Find the equation of a line perpendicular to a given line and containing a given point.
- Find the midpoint of a line segment.
- Use the completing-the-square technique with quadratic expressions.
- Solve quadratic equations.
- Find the equation of a circle, given its center and radius.
- Find the vertex of a parabola.
- Compute the area of triangles and trapezoids.
- Compute the area inside a circle or ellipse.
- Explain how area changes when stretching either horizontally or vertically or both.

To review a chapter, go through the list above to find items that you do not know how to do, then reread the material in the chapter about those items. Then try to answer the chapter review questions below without looking back at the chapter.

CHAPTER REVIEW QUESTIONS

1. Find the distance between the points (5, −6) and (−2, −4).
2. Find two points, one on the horizontal axis and one on the vertical axis, such that the distance between these two points equals 21.
3. Find the perimeter of the parallelogram whose vertices are (2, 1), (7, 1), (10, 3), and (5, 3).
4. Find the perimeter of the triangle whose vertices are (1, 2), (6, 2), and (7, 5).
5. Find the perimeter of the trapezoid whose vertices are (2, 3), (8, 3), (9, 5), and (−1, 5).
6. Explain how to find the slope of a line if given the coordinates of two points on the line.
7. Given the slopes of two lines, how can you determine whether or not the lines are parallel?
8. Given the slopes of two lines, how can you determine whether or not the lines are perpendicular?
9. Find a number t such that the line containing the points (3, −5) and (−4, t) has slope −6.
10. Find the equation of the line in the xy-plane that has slope −4 and contains the point (3, −7).
11. Find the equation of the line in the xy-plane that contains the points (−6, 1) and (−1, −8).
12. Find the equation of the line in the xy-plane that is perpendicular to the line \(y = 6x − 7\) and that contains the point (−2, 9).
13. Find a line segment that is not parallel to either of the coordinate axes and that has (−3, 5) as its midpoint.
14. Find the vertex of the graph of the equation \(y = 5x^2 + 2x + 3\).
15. Give an example of numbers a, b, and c such that the graph of \(y = ax^2 + bx + c\) has its vertex at the point (−4, 7).
16. Find a number $c$ such that the equation
\[ x^2 + cx + 3 = 0 \]
has exactly one solution.

17. Find a number $x$ such that
\[ \frac{x + 1}{x - 2} = 3x. \]

18. Find the equation of the circle in the $xy$-plane centered at $(-4, 3)$ that has radius 6.

19. Find the center, radius, and circumference of the circle in the $xy$-plane described by
\[ x^2 - 8x + y^2 + 10y = 2. \]
Also, find the area inside this circle.

20. Find the area of a triangle that has two sides of length 8 and one side of length 3.

21. Find the area of the parallelogram whose vertices are $(2, 1), (7, 1), (10, 3),$ and $(5, 3)$.

22. Find the area of the triangle whose vertices are $(1, 2), (6, 2),$ and $(7, 5)$.

23. Find the area of the trapezoid whose vertices are $(2, 3), (8, 3), (9, 5),$ and $(-1, 5)$.

24. Find a number $t$ such that the area inside the circle
\[ x^2 + 6x + y^2 - 8y = t \]
is 11.

25. Find the area inside the ellipse
\[ 3x^2 + 2y^2 = 5. \]

26. Suppose a newly discovered planet is orbiting a far-away star and that units and a coordinate system have been chosen so that planet's orbit is described by the equation
\[ \frac{x^2}{29} + \frac{y^2}{20} = 1. \]
What are the two possible locations of the star?