CHAPTER

2

René Descartes, who invented the coordinate system we use to graph equations, explaining his work to Queen Christina of Sweden (from an 18^{th}-century painting by Dumesnil).

Combining Algebra and Geometry

Analytic geometry, which combines algebra and geometry, provides a tremendously powerful tool for visualizing equations with two variables.

We begin this chapter with a description of the coordinate plane, including a discussion of distance and length. Then we turn our attention to lines and their slopes, which are simple concepts that have immense importance.

We then investigate quadratic expressions. We will see how to complete the square and solve quadratic equations. Quadratic expressions will also lead us to the conic sections (ellipses, parabolas, and hyperbolas).

We conclude the chapter by learning methods for computing the area of triangles, trapezoids, circles, and ellipses.

2.1 *The Coordinate Plane*

LEARNING OBJECTIVES

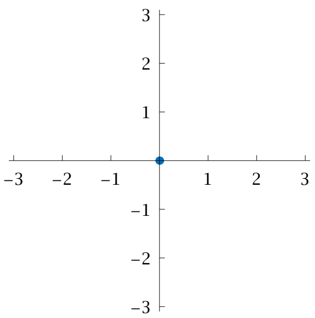
By the end of this section you should be able to

- locate points in the coordinate plane;
- graph equations with two variables in the coordinate plane, possibly using technology;
- compute the distance between two points;
- compute the circumference of a circle.

Coordinates

Recall how the real line is constructed: We start with a horizontal line, pick a point on it that we label 0, pick a point to the right of 0 that we label 1, and then we label other points using the scale determined by 0 and 1 (see Section 1.1 to review the construction of the real line).

The coordinate plane is constructed in a similar fashion, but using a horizontal and a vertical line rather than just a horizontal line.



The coordinate plane, with a dot at the origin.

The coordinate plane

- The **coordinate plane** is constructed by starting with a horizontal line and a vertical line in a plane. These lines are called the **coordinate axes.**
- The intersection point of the coordinate axes is called the **origin**; it receives a label of 0 on both axes.
- On the horizontal axis, pick a point to the right of the origin and label it 1. Then label other points on the horizontal axis using the scale determined by the origin and 1.
- Similarly, on the vertical axis, pick a point above the origin and label it 1. Then label other points on the vertical axis using the scale determined by the origin and 1.

Sometimes it is important to use the same scale on both axes, as is done in the figure above. Other times it may be more convenient to use two different scales on the two axes.

A point in the plane is identified with its coordinates, which are written as an ordered pair of numbers surrounded by parentheses, as described below.

Coordinates

- The first coordinate indicates the horizontal distance from the origin, with positive numbers corresponding to points right of the origin and negative numbers corresponding to points left of the origin.
- The second coordinate indicates the vertical distance from the origin, with positive numbers corresponding to points above the origin and negative numbers corresponding to points below the origin.

The plane with this system of labeling is often called the **Cartesian plane** *in honor of the French mathematician René Descartes (1596–1650), who described this technique in his 1637 book* Discourse on Method.

EXAMPLE 1

Locate on a coordinate plane the following points:

(a) $(2, 1)$; (b) $(-1, 2.5)$; (c) $(-2.5, -2.5)$; (d) $(3, -2)$.

SOLUTION

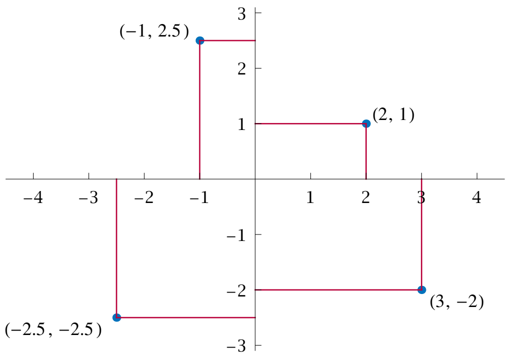
(a) The point $(2, 1)$ can be located by starting at the origin, moving 2 units to the right along the horizontal axis, and then moving up 1 unit; see the figure below.

(b) The point $(-1, 2.5)$ can be located by starting at the origin, moving 1 unit to the left along the horizontal axis, and then moving up 2.5 units; see the figure below.

(c) The point $(-2.5, -2.5)$ can be located by starting at the origin, moving 2.5 units to the left along the horizontal axis, and then moving down 2.5 units; see the figure below.

(d) The point $(3, -2)$ can be located by starting at the origin, moving 3 units to the right along the horizontal axis, and then moving down 2 units; see the figure below.

The notation $(-1, 2.5)$ *could denote either the point with coordinates* $(-1, 2.5)$ *or the open interval* $(-1, 2.5)$. *You should be able to tell from the context which meaning is intended.*



These coordinates are sometimes called **rectangular coordinates** *because each point's coordinates are determined by a rectangle, as shown in this figure.*

The horizontal axis is often called the x**-axis** and the vertical axis is often called the y**-axis.** In this case, the coordinate plane can be called the xy-plane. However, other variables can also be used, depending on the problem at hand.

Regardless of the names of the axes, remember that the first coordinate corresponds to horizontal distance from the origin and the second coordinate corresponds to vertical distance from the origin.

If the horizontal axis has been labeled the x-axis, then the first coordinate of a point is often called the x-**coordinate**. Similarly, if the vertical axis has been labeled the y-axis, then the second coordinate is often called the y-**coordinate**.

The potential confusion of this terminology becomes apparent when we want to consider a point whose coordinates are (y, x); here y is the x-coordinate and x is the y-coordinate. Furthermore, always calling the first coordinate the x-coordinate will lead to confusion when the horizontal axis is labeled with another variable such as t or θ.

Graphs of Equations

The coordinate plane allows us to visualize the set of points satisfied by equations with two variables.

Graph of an equation

The **graph** of an equation with two variables is the set of points in the corresponding coordinate plane satisfied by the equation.

EXAMPLE 2

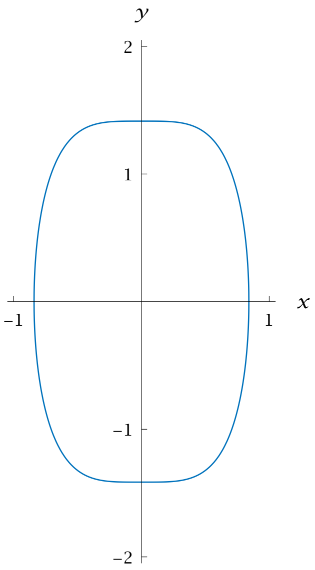
The graph of the equation $4x^4 + y^2 = 2$ is shown in the margin.

(a) Where does this graph intersect the x-axis?

(b) Where does this graph intersect the y-axis?

(c) Is the point $(\frac{1}{2}, \frac{7}{5})$ on this graph?

(d) Find four points on this graph that are not on either of the coordinate axes.



The graph of the equation $4x^4 + y^2 = 2$.

SOLUTION

(a) The x-axis is the set of points where $y = 0$. Thus to find the points where the graph intersects the x-axis, substitute $y = 0$ in the equation $4x^4 + y^2 = 2$ to obtain the equation $4x^4 = 2$, which implies that $x^4 = \frac{1}{2}$, which implies that $x = 1/2^{1/4} \approx 0.8$ or $x = -1/2^{1/4} \approx -0.8$ (we will discuss fractional exponents in detail in Section 5.1, but you are probably already familiar with this concept from a previous course).

Thus the points where the graph intersects the x-axis are approximately $(0.8, 0)$ and $(-0.8, 0)$. Make sure you can locate these points on the graph in the margin.

(b) The y-axis is the set of points where $x = 0$. Thus to find the points where the graph intersects the y-axis, substitute $x = 0$ in the equation $4x^4 + y^2 = 2$ to obtain the equation $y^2 = 2$, which implies that $y = \sqrt{2} \approx 1.4$ or $y = -\sqrt{2} \approx -1.4$.

Thus the points where the graph intersects the y-axis are approximately $(0, 1.4)$ and $(0, -1.4)$. Make sure you can locate these points on the graph in the margin.

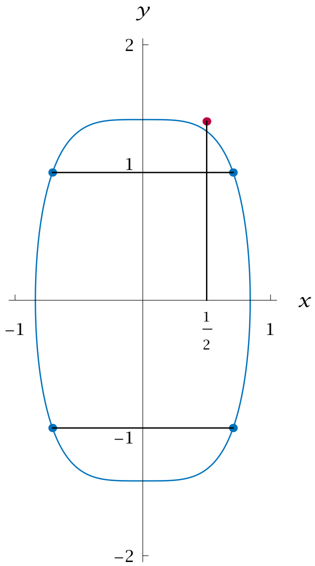
(c) Asking whether $(\frac{1}{2}, \frac{7}{5})$ is on the graph of the equation $4x^4 + y^2 = 2$ is equivalent to asking whether $4(\frac{1}{2})^4 + (\frac{7}{5})^2$ equals 2. A little arithmetic shows that $4(\frac{1}{2})^4 + (\frac{7}{5})^2$ equals $\frac{221}{100}$, which equals 2.21, which is close but not equal to 2. Thus the point $(\frac{1}{2}, \frac{7}{5})$ is not on the graph of $4x^4 + y^2 = 2$.

The red dot in the figure in the margin shows the point $(\frac{1}{2}, \frac{7}{5})$. As you can see, it is not on the graph of $4x^4 + y^2 = 2$, although it is close. The vertical line segment has endpoints at $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{7}{5})$, showing that the red dot has first coordinate equal to $\frac{1}{2}$.

(d) To find some points on the graph not on the coordinate axes, substitute $y = 1$ in the equation $4x^4 + y^2 = 2$ to obtain the equation $4x^4 = 1$, which implies that $x = 1/4^{1/4} \approx 0.7$ or $x = -1/4^{1/4} \approx -0.7$. Thus $(1/4^{1/4}, 1)$ and $(-1/4^{1/4}, 1)$, which are approximately $(0.7, 1)$ and $(-0.7, 1)$, are points on the graph.

Because $(-y)^2 = y^2$, the points $(1/4^{1/4}, -1)$ and $(-1/4^{1/4}, -1)$, which are approximately $(0.7, -1)$ and $(-0.7, -1)$, are also on the graph.

The blue dots in the figure in the margin show the four points on the graph that we have just found. The top horizontal line segment has endpoints at $(-1/4^{1/4}, 1)$ and $(1/4^{1/4}, 1)$, showing that these two points have second coordinate equal to 1. The bottom horizontal line segment has endpoints at $(-1/4^{1/4}, -1)$ and $(1/4^{1/4}, -1)$, showing that these two points have second coordinate equal to -1.



The graph of the equation $4x^4 + y^2 = 2$, *along with one point not on the graph and four points on the graph.*

Later in this chapter we will discuss in detail graphs that are lines, circles, ellipses, parabolas, and hyperbolas. Throughout the book we will discuss the graphs of a large variety of functions that we will study. Meanwhile, use Wolfram|Alpha or a graphing calculator to experiment with the graphs of equations.

The next example uses Wolfram|Alpha in the solution, but you could use a graphing calculator or any other technology instead.

EXAMPLE 3

(a) Graph the equation $2t^3 + z = z^3 + t$.

(b) Graph the equation $2t^3 + z = z^3 + t$ for t in the interval $[-6, 6]$.

(c) Determine whether or not the point $(4.5, 5.5)$ is on the graph of $2t^3 + z = z^3 + t$.

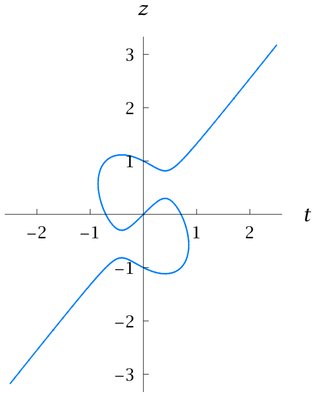
SOLUTION

(a) Point a web browser to `www.wolframalpha.com`. In the one-line entry box that appears near the top of the web page, enter

```
graph 2t^3 + z = z^3 + t
```

and then press the *enter* key on your keyboard or click the = box on the right side of the Wolfram|Alpha entry box, getting a graph that should look like the one shown in the margin here.

Wolfram|Alpha selects the variable lowest in alphabetical order to use as the horizontal axis and the variable highest in alphabetical order to use as the vertical axis, as has been done here with t and z and as is usually what you want to do when the variables are x and y.



The graph of the equation $2t^3 + z = z^3 + t$.

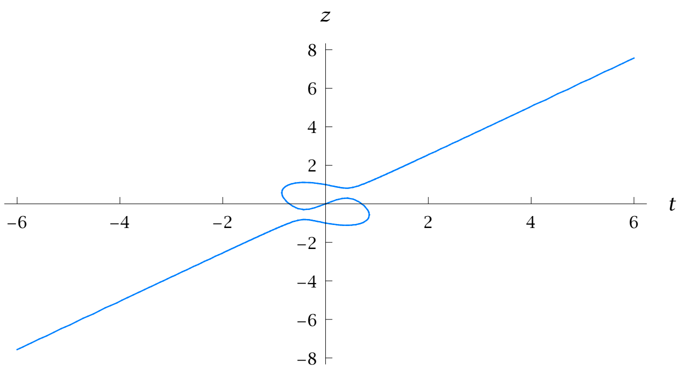
(b) Unlike the previous graph we examined, the graph of $2t^3 + z = z^3 + t$ includes points arbitrarily far left, right, up, and down. Thus the figure shown in the margin above is not the entire graph of $2t^3 + z = z^3 + t$. No figure could show the entire graph of this equation.

Wolfram|Alpha usually chooses an interesting part of the graph to show, but you may want to zoom in to see more detail or zoom out to see a bigger picture. For example, to zoom out to see the graph for t in the interval $[-6, 6]$, type

graph 2t^3 + z = z^3 + t from t =-6 to 6

in a Wolfram|Alpha entry box to obtain the graph shown below.

Here the horizontal and vertical axes have different scales, which can distort the shape of the graph. However, the use of different scales is often necessary to prevent the graph from becoming too large.



The graph of $2t^3 + z = z^3 + t$ *for* t *in the interval* $[-6, 6]$.

(c) From the graph above, it appears possible that the point $(4.5, 5.5)$ may be on the graph of $2t^3 + z = z^3 + t$. To check whether or not this is true, we substitute $t = 4.5$ and $z = 5.5$ into the equation and check whether we get a true statement. To do this check using Wolfram|Alpha, type

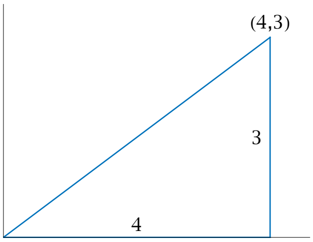
is 2 * 4.5^3 + 5.5 = 5.5^3 + 4.5

in a Wolfram|Alpha entry box to obtain the answer False. Thus the point $(4.5, 5.5)$ is not on the graph of $2t^3 + z = z^3 + t$.

Distance Between Two Points

We start gently with some concrete examples before getting to the formula for the distance between two points.

EXAMPLE 4 Find the distance between the point $(4, 3)$ and the origin.

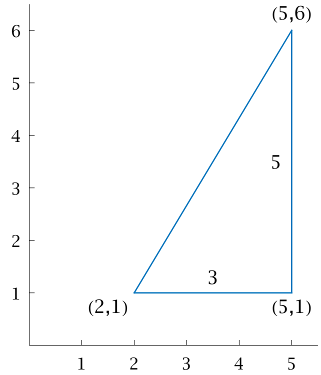


SOLUTION The distance between the point $(4, 3)$ and the origin is the length of the hypotenuse in the right triangle shown here. By the Pythagorean Theorem, this hypotenuse has length $\sqrt{4^2 + 3^2}$, which equals $\sqrt{25}$, which equals 5.

Here is another example, this time with neither of the points being the origin.

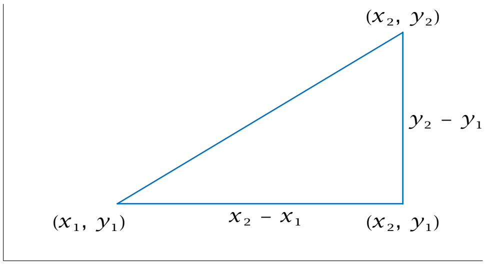
EXAMPLE 5

Find the distance between the points $(5,6)$ and $(2,1)$.



SOLUTION The distance between the points $(5,6)$ and $(2,1)$ is the length of the hypotenuse in the right triangle shown here. The horizontal side of this triangle has length $5-2$, which equals 3, and the vertical side of this triangle has length $6-1$, which equals 5. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{3^2+5^2}$, which equals $\sqrt{34}$.

More generally, to find the formula for the distance between two points (x_1,y_1) and (x_2,y_2), consider the right triangle in the figure below:



The length of the hypotenuse equals the distance between (x_1,y_1) and (x_2,y_2).

Starting with the points (x_1,y_1) and (x_2,y_2) in the figure above, make sure you understand why the third point in the triangle (the vertex at the right angle) has coordinates (x_2,y_1). Also, verify that the horizontal side of the triangle has length x_2-x_1 and the vertical side of the triangle has length y_2-y_1, as indicated in the figure above. The Pythagorean Theorem then gives the length of the hypotenuse, leading to the following formula:

Distance between two points

The distance between the points (x_1,y_1) and (x_2,y_2) is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

As a special case of this formula, the distance between a point (x,y) and the origin is $\sqrt{x^2+y^2}$.

Using the formula above, we can now find the distance between two points without drawing a figure.

EXAMPLE 6

Find the distance between the points $(3,1)$ and $(-4,-99)$.

SOLUTION The distance between these two points is $\sqrt{(3-(-4))^2+(1-(-99))^2}$, which equals $\sqrt{7^2+100^2}$, which equals $\sqrt{49+10000}$, which equals $\sqrt{10049}$.

Length, Perimeter, and Circumference

The **length** of a line segment is the distance between the two endpoints. For example, the length of the line segment connecting the points $(-1,4)$ and $(2,6)$ equals

$$\sqrt{\big(2-(-1)\big)^2+(6-4)^2},$$

which equals $\sqrt{13}$.

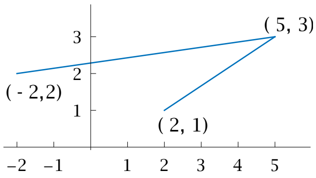
Defining the length of a path or curve in the coordinate plane is more complicated. A rigorous definition requires calculus, so we use the following intuitive definition:

Length

The **length** of a path or curve can be determined by placing a string on the path or curve and then measuring the length of the string when it is straightened into a line segment.

EXAMPLE 7

Find the length of the path shown here consisting of the line segment connecting $(-2,2)$ with $(5,3)$ followed by the line segment connecting $(5,3)$ with $(2,1)$.



If a path consists of line segments, then the length of the path is the sum of the lengths of the line segments.

SOLUTION The first line segment has length

$$\sqrt{\big(5-(-2)\big)^2+(3-2)^2},$$

which equals $\sqrt{50}$. The second line segment has length

$$\sqrt{(2-5)^2+(1-3)^2},$$

which equals $\sqrt{13}$. Thus this path has length $\sqrt{50}+\sqrt{13}$.

You are probably already familiar with two other words that are used to denote the lengths of certain paths that begin and end at the same point. One word probably would have been enough, but the two following words are commonly used:

Perimeter and circumference

- The **perimeter** of a polygon is the length of the path that surrounds the polygon.
- The **circumference** of a region is the length of the curve that surrounds the region.

EXAMPLE 8

(a) What is the perimeter of an equilateral triangle with sides of length s?

(b) What is the perimeter of a square with sides of length s?

(c) What is the perimeter of a rectangle with width w and height h?

SOLUTION

(a) An equilateral triangle with sides of length s has perimeter $3s$.

(b) A square with sides of length s has perimeter $4s$.

(c) A rectangle with width w and height h has perimeter $2w + 2h$.

Just for fun, here are the first 504 digits of π:

3.14159265358979323
846264338327950288
419716939937510582
097494459230781640
628620899862803482
534211706798214808
651328230664709384
460955058223172535
940812848111745028
410270193852110555
964462294895493038
196442881097566593
344612847564823378
678316527120190914
564856692346034861
045432664821339360
726024914127372458
700660631558817488
152092096282925409
171536436789259036
001133053054882046
652138414695194151
160943305727036575
959195309218611738
193261179310511854
807446237996274956
735188575272489122
793818301194912983

The perimeter of an equilateral triangle is proportional to the length of one of its sides (the ratio is 3) and the perimeter of a square is proportional to the length of one of its sides (the ratio is 4). Thus it is reasonable to believe that the circumference of a circle is proportional to its diameter.

Physical experiments confirm this belief. For example, suppose you have a very accurate ruler that can measure lengths with 0.01-inch accuracy. If you place a string on top of a circle with diameter 1 inch, then straighten the string to a line segment, you will find that the string has length about 3.14 inches. Similarly, if you place a string on top of a circle with diameter 2 inches, then straighten the string to a line segment, you will find that the string has length about 6.28 inches. Thus the circumference of a circle with diameter two inches is twice the circumference of a circle with diameter 1 inch.

The circle on the left has been straightened into the line segment on the right. A measurement shows that this line segment is approximately 3.14 *times as long as the diameter of the circle, which is shown above in red.*

Similarly, you will find that for any circle you measure, the ratio of the circumference to the diameter is approximately 3.14. The exact value of this ratio is so important that it gets its own symbol:

> π
>
> The ratio of the circumference to the diameter of a circle is called π.

It turns out that π is an irrational number. For most practical purposes, 3.14 is a good approximation of π—the error is about 0.05%. If more accurate computations are needed, then 3.1416 is an even better approximation—the error is about 0.0002%.

Does the decimal expansion of π contain one thousand consecutive 4's? No one knows, but mathematicians suspect that the answer is "yes".

The remarkable approximation $\pi \approx \frac{355}{113}$ was discovered over 1500 years ago by the Chinese mathematician Zu Chongzhi.

A fraction that approximates π well is $\frac{22}{7}$ (notice how page 22 is numbered in this book)—the error is about 0.04%. A fraction that approximates π even better is $\frac{355}{113}$—the error is extremely small at about 0.000008%.

Keep in mind that π is not equal to 3.14 or 3.1416 or $\frac{22}{7}$ or $\frac{355}{113}$. All of these are useful approximations, but π is an irrational number that cannot be represented exactly as a decimal number or as a fraction.

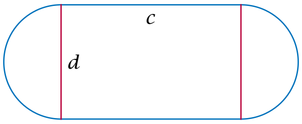
We have defined π to be the number such that a circle with diameter d has circumference πd. Because the diameter of a circle is equal to twice the radius, we have the following formula:

Circumference of a circle

A circle with radius r has circumference $2\pi r$.

EXAMPLE 9

Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal the total length of the straight part of the track. What dimensions should the track have?

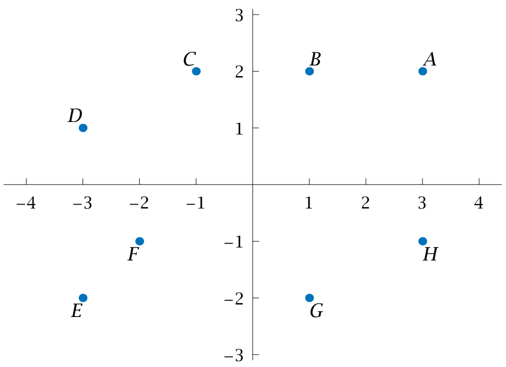


SOLUTION We want the total length of the straight part of the track to be 200 meters. Thus each of the two straight pieces must be 100 meters long. Hence we take $c = 100$ meters in the figure in the margin.

We also want the total length of the two half-circles to be 200 meters. Thus we want $\pi d = 200$. Hence we take $d = \frac{200}{\pi} \approx 63.66$ meters in the figure in the margin.

EXERCISES

For Exercises 1–8, give the coordinates of the specified point using the figure below:



1. A	3. C	5. E	7. G
2. B	4. D	6. F	8. H

9. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(1, 2)$, $(-2, 2)$, $(-3, -1)$, $(2, -3)$.

10. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(2.5, 1)$, $(-1, 3)$, $(-1.5, -1.5)$, $(1, -3)$.

11. Find the distance between the points $(3, -2)$ and $(-1, 4)$.

12. Find the distance between the points $(-4, -7)$ and $(-8, -5)$.

13. Find two choices for t such that the distance between $(2, -1)$ and $(t, 3)$ equals 7.

14. Find two choices for t such that the distance between $(3, -2)$ and $(1, t)$ equals 5.

15. Find two choices for b such that $(4, b)$ has distance 5 from $(3, 6)$.

16. Find two choices for b such that $(b, -1)$ has distance 4 from $(3, 2)$.

17. Find two points on the horizontal axis whose distance from $(3, 2)$ equals 7.

18. Find two points on the horizontal axis whose distance from $(1, 4)$ equals 6.

19. A ship sails north for 2 miles and then west for 5 miles. How far is the ship from its starting point?

20. A ship sails east for 7 miles and then south for 3 miles. How far is the ship from its starting point?

Use the following information concerning Manhattan (New York City) for Exercises 21–22:

- ***Avenues in Manhattan run roughly north-south; streets run east-west.***
- ***For much of its length, Broadway runs diagonally across the grid formed by avenues and streets.***
- ***The distance between consecutive avenues in Manhattan is 922 feet.***
- ***The distance between consecutive streets in Manhattan is 260 feet.***

21. Suppose you walk from the corner of Central Park at 8th Avenue and 59th Street to 10th Avenue and 71st Street.

    (a) What is the length of your path if you walk along 8th Avenue to 71st Street, then along 71st Street to 10th Avenue?

    (b) What is the length of your path if you walk along Broadway, which goes in a straight line from 8th Avenue and 59th Street to 10th Avenue and 71st Street?

    (c) How much shorter is the direct path along Broadway than walking along 8th Avenue and 71st Street?

    (d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 8th Avenue and 71st Street?

22. Suppose you walk from 5th Avenue and 24th Street to Times Square at 7th Avenue and 42nd Street.

    (a) What is the length of your path if you walk along 5th Avenue to 42nd Street, then along 42nd Street to 7th Avenue?

    (b) What is the length of your path if you walk along Broadway, which goes in a straight line from 5th Avenue and 24th Street to 7th Avenue and 42nd Street?

    (c) How much shorter is the direct path along Broadway than walking along 5th Avenue and 42nd Street?

    (d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 5th Avenue and 42nd Street?

23. Find two points on the vertical axis whose distance from $(5, -1)$ equals 8.

24. Find two points on the vertical axis whose distance from $(2, -4)$ equals 5.

25. Find the perimeter of the triangle that has vertices at $(1, 2)$, $(5, -3)$, and $(-4, -1)$.

26. Find the perimeter of the triangle that has vertices at $(-3, 1)$, $(4, -2)$, and $(5, -1)$.

27. Find the radius of a circle that has circumference 12 inches.

28. Find the radius of a circle that has circumference 20 feet.

29. Find the radius of a circle that has circumference 8 more than its diameter.

30. Find the radius of a circle that has circumference 12 more than its diameter.

31. Find an equation whose graph in the xy-plane is the set of points whose distance from the origin is 3.

32. Find an equation whose graph in the xy-plane is the set of points whose distance from the origin is 5.

33. Find the length of the graph of the equation $x^2 + y^2 = 9$.

34. Find the length of the graph of the equation $x^2 + y^2 = 25$.

35. Find an equation whose graph in the rt-plane is the set of points whose distance from $(3, 2)$ is 2.

36. Find an equation whose graph in the bc-plane is the set of points whose distance from $(-2, 1)$ is 3.

37. Find two points that have distance 2 from the origin and distance 3 from $(3, 0)$.

38. Find two points that have distance 3 from the origin and distance 2 from $(2, 0)$.

39. Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal half the total length of the straight part of the track. What dimensions should the track have?

40. Suppose you want to design a 200-meter indoor track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal three-fourths the total length of the straight part of the track. What dimensions should the track have?

41. Suppose a rope is just long enough to cover the equator of the Earth. About how much longer would the rope need to be so that it could be suspended seven feet above the entire equator?

42. Suppose a satellite is in orbit one hundred miles above the equator of the Earth. About how much further does the satellite travel in one orbit than would a person traveling once around the equator on the surface of the Earth?

PROBLEMS

Some problems require considerably more thought than the exercises. Unlike exercises, problems often have more than one correct answer.

43. Find two points, one on the horizontal axis and one on the vertical axis, such that the distance between these two points equals 15.

44. Explain why there does not exist a point on the horizontal axis whose distance from $(5, 4)$ equals 3.

45. Use Wolfram|Alpha or a calculator to find the distance between the points $(-21, -15)$ and $(17, 28)$. [In Wolfram|Alpha, you can do this by typing `distance from (-21, -15) to (17, 28)` in an entry box. Note that in addition to the distance in both exact and approximate form, you get a figure showing the two points. Experiment with finding the distance between other pairs of points, and notice the placement of the points on the figures produced by Wolfram|Alpha.]

46. Find six distinct points whose distance from the origin equals 3.

47. Find six distinct points whose distance from $(3, 1)$ equals 4.

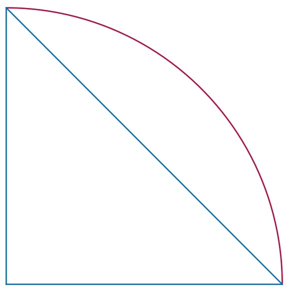
48. Graph the equation $x^4 + y^4 = 1$.

49. (a) Graph the equation $y^3 - 3y = x^2$ for x in the interval $[-3, 3]$.

    (b) What is unusual about this graph as compared to other graphs we have examined?

50. Show that a square whose diagonal has length d has perimeter $2\sqrt{2}d$.

51. The figure below illustrates an isosceles right triangle with legs of length 1, along with one-fourth of a circle centered at the right-angle vertex of the triangle. Using the result that the shortest path between two points is a line segment, explain why this figure shows that $2\sqrt{2} < \pi$.

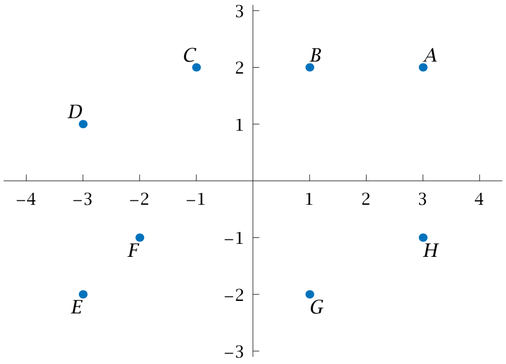


WORKED-OUT SOLUTIONS *to Odd-numbered Exercises*

Do not read these worked-out solutions before first attempting to do the exercises yourself. Otherwise you may merely mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises (even if they have not been assigned) and check your answers here. If you get stuck on an exercise, reread the section of the textbook—then try the exercise again. If you are still stuck, then look at the worked-out solution here.

For Exercises 1–8, give the coordinates of the specified point using the figure below:



1. A

SOLUTION To get to the point A starting at the origin, we must move 3 units right and 2 units up. Thus A has coordinates $(3, 2)$.

Numbers obtained from a figure should be considered approximations. Thus the actual coordinates of A might be $(3.01, 1.98)$.

3. C

SOLUTION To get to the point C starting at the origin, we must move 1 unit left and 2 units up. Thus C has coordinates $(-1, 2)$.

5. E

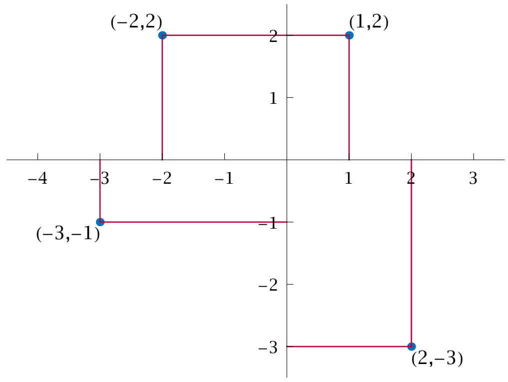
SOLUTION To get to the point E starting at the origin, we must move 3 units left and 2 units down. Thus E has coordinates $(-3, -2)$.

7. G

SOLUTION To get to the point G starting at the origin, we must move 1 unit right and 2 units down. Thus G has coordinates $(1, -2)$.

9. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(1, 2)$, $(-2, 2)$, $(-3, -1)$, $(2, -3)$.

SOLUTION



11. Find the distance between the points $(3, -2)$ and $(-1, 4)$.

SOLUTION The distance between the points $(3, -2)$ and $(-1, 4)$ equals

$$\sqrt{(-1-3)^2 + \left(4-(-2)\right)^2},$$

which equals $\sqrt{(-4)^2 + 6^2}$, which equals $\sqrt{16+36}$, which equals $\sqrt{52}$, which can be simplified as follows:

$$\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}.$$

Thus the distance between the points $(3, -2)$ and $(-1, 4)$ equals $2\sqrt{13}$.

13. Find two choices for t such that the distance between $(2, -1)$ and $(t, 3)$ equals 7.

SOLUTION The distance between $(2, -1)$ and $(t, 3)$ equals

$$\sqrt{(t-2)^2 + 16}.$$

We want this to equal 7, which means that we must have

$$(t-2)^2 + 16 = 49.$$

Subtracting 16 from both sides of the equation above gives

$$(t-2)^2 = 33,$$

which implies that $t - 2 = \pm\sqrt{33}$. Thus $t = 2 + \sqrt{33}$ or $t = 2 - \sqrt{33}$.

15. Find two choices for b such that $(4, b)$ has distance 5 from $(3, 6)$.

SOLUTION The distance between $(4, b)$ and $(3, 6)$ equals

$$\sqrt{1 + (6-b)^2}.$$

We want this to equal 5, which means that we must have

$$1 + (6-b)^2 = 25.$$

Subtracting 1 from both sides of the equation above gives

$$(6-b)^2 = 24,$$

which implies that $6 - b = \pm\sqrt{24}$. Thus $b = 6 - \sqrt{24}$ or $b = 6 + \sqrt{24}$.

17. Find two points on the horizontal axis whose distance from $(3, 2)$ equals 7.

SOLUTION A typical point on the horizontal axis has coordinates $(x, 0)$. The distance from this point to $(3, 2)$ is $\sqrt{(x-3)^2 + (0-2)^2}$. Thus we need to solve the equation

$$\sqrt{(x-3)^2 + 4} = 7.$$

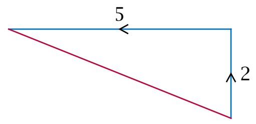
Squaring both sides of the equation above, and then subtracting 4 from both sides gives

$$(x-3)^2 = 45.$$

Thus $x - 3 = \pm\sqrt{45} = \pm 3\sqrt{5}$. Thus $x = 3 \pm 3\sqrt{5}$. Hence the two points on the horizontal axis whose distance from $(3, 2)$ equals 7 are $(3 + 3\sqrt{5}, 0)$ and $(3 - 3\sqrt{5}, 0)$.

19. A ship sails north for 2 miles and then west for 5 miles. How far is the ship from its starting point?

SOLUTION The figure below shows the path of the ship. The length of the red line is the distance of the ship from its starting point. By the Pythagorean Theorem, this distance is $\sqrt{2^2 + 5^2}$ miles, which equals $\sqrt{29}$ miles.



We have assumed that the surface of the Earth is part of a plane rather than part of a sphere. For distances of less than a few hundred miles, this is a good approximation.

Use the following information concerning Manhattan (New York City) for Exercises 21–22:

- ***Avenues in Manhattan run roughly north-south; streets run east-west.***
- ***For much of its length, Broadway runs diagonally across the grid formed by avenues and streets.***
- ***The distance between consecutive avenues in Manhattan is 922 feet.***
- ***The distance between consecutive streets in Manhattan is 260 feet.***

21. Suppose you walk from the corner of Central Park at 8th Avenue and 59th Street to 10th Avenue and 71st Street.

(a) What is the length of your path if you walk along 8th Avenue to 71st Street, then along 71st Street to 10th Avenue?

(b) What is the length of your path if you walk along Broadway, which goes in a straight line from 8th Avenue and 59th Street to 10th Avenue and 71st Street?

(c) How much shorter is the direct path along Broadway than walking along 8th Avenue and 71st Street?

(d) At the normal city walking speed of 250 feet per minute, how much time would you save by walking along Broadway as compared to walking along 8th Avenue and 71st Street?

SOLUTION

(a) Walking along 8th Avenue from 59th Street to 71st Street is 12 blocks (because $71 - 59 = 12$), each of which is 260 feet, for a total of 12×260 feet, which equals 3120 feet.

Walking along 71st Street from 8th Avenue to 10th Avenue is 2 blocks, each of which is 922 feet, for a total of 1844 feet.

Thus the total path consists of 3120 feet plus 1844 feet, which equals 4964 feet.

(b) The length of the path along Broadway is the length of the hypotenuse of a right triangle whose other sides have length 3120 feet and 1844 feet, as calculated in part (a). Thus the length of the path along Broadway is

$$\sqrt{3120^2 + 1844^2} = \sqrt{13134736} \approx 3624 \text{ feet.}$$

(c) Subtracting the results of part (b) from the result of part (a), we see that the path along Broadway is about 1340 feet shorter. Because a mile contains 5280 feet, the path along Broadway is about a quarter-mile shorter ($\frac{1340}{5280} \approx 0.25$).

(d) Part (c) shows that the path along Broadway is about 1340 feet shorter. Thus the amount of time saved is $\frac{1340}{250}$ minutes, which equals a bit more than 5 minutes.

23. Find two points on the vertical axis whose distance from $(5, -1)$ equals 8.

SOLUTION A typical point on the vertical axis has coordinates $(0, y)$. The distance from this point to $(5, -1)$ is $\sqrt{(0-5)^2 + \big(y - (-1)\big)^2}$. Thus we need to solve the equation

$$\sqrt{25 + (y+1)^2} = 8.$$

Squaring both sides of the equation above, and then subtracting 25 from both sides gives

$$(y+1)^2 = 39.$$

Thus $y + 1 = \pm\sqrt{39}$. Thus $y = -1 \pm \sqrt{39}$. Hence the two points on the vertical axis whose distance from $(5, -1)$ equals 8 are $(0, -1 + \sqrt{39})$ and $(0, -1 - \sqrt{39})$.

25. Find the perimeter of the triangle that has vertices at $(1, 2)$, $(5, -3)$, and $(-4, -1)$.

SOLUTION The perimeter of the triangle equals the sum of the lengths of the three sides of the triangle. Thus we find the lengths of those three sides.

The side of the triangle connecting the vertices $(1, 2)$ and $(5, -3)$ has length

$$\sqrt{(5-1)^2 + (-3-2)^2} = \sqrt{41}.$$

The side of the triangle connecting the vertices $(5, -3)$ and $(-4, -1)$ has length

$$\sqrt{(-4-5)^2 + \big(-1-(-3)\big)^2} = \sqrt{85}.$$

The side of the triangle connecting the vertices $(-4, -1)$ and $(1, 2)$ has length

$$\sqrt{\big(1-(-4)\big)^2 + \big(2-(-1)\big)^2} = \sqrt{34}.$$

Thus the perimeter of the triangle equals $\sqrt{41} + \sqrt{85} + \sqrt{34}$.

27. Find the radius of a circle that has circumference 12 inches.

SOLUTION Let r denote the radius of this circle in inches. Thus $2\pi r = 12$, which implies that $r = \frac{6}{\pi}$ inches.

29. Find the radius of a circle that has circumference 8 more than its diameter.

SOLUTION Let r denote the radius of this circle. Thus the circle has circumference $2\pi r$ and has diameter $2r$. Because the circumference is 8 more than diameter, we have $2\pi r = 2r + 8$. Thus $(2\pi - 2)r = 8$, which implies that $r = \frac{4}{\pi - 1}$.

31. Find an equation whose graph in the xy-plane is the set of points whose distance from the origin is 3.

SOLUTION The distance from a point (x, y) to the origin is $\sqrt{x^2 + y^2}$. Thus the equation we seek is

$$\sqrt{x^2 + y^2} = 3.$$

To write this equation without using square roots, square both sides to get the equivalent equation

$$x^2 + y^2 = 9.$$

33. Find the length of the graph of the equation $x^2 + y^2 = 9$.

SOLUTION From Exercise 31, we see that the graph of $x^2 + y^2 = 9$ is the set of points in the xy-plane whose distance from the origin is 3. In other words, the graph of $x^2 + y^2 = 9$ is the circle of radius 3 centered at the origin. The length (or circumference) of this graph is $2\pi \cdot 3$, which equals 6π.

35. Find an equation whose graph in the rt-plane is the set of points whose distance from $(3, 2)$ is 2.

SOLUTION The distance from a point (r, t) to $(3, 2)$ is $\sqrt{(r-3)^2 + (t-2)^2}$. Thus the equation we seek is

$$\sqrt{(r-3)^2 + (t-2)^2} = 2.$$

To write this equation without using square roots, square both sides to get the equivalent equation

$$(r-3)^2 + (t-2)^2 = 4.$$

37. Find two points that have distance 2 from the origin and distance 3 from $(3, 0)$.

SOLUTION Suppose (x, y) has distance 2 from the origin and distance 3 from $(3, 0)$. Thus

$$\sqrt{x^2 + y^2} = 2 \quad \text{and} \quad \sqrt{(x-3)^2 + y^2} = 3.$$

Squaring both sides of these equations gives

$$x^2 + y^2 = 4 \quad \text{and} \quad x^2 - 6x + 9 + y^2 = 9.$$

Subtracting the second equation from the first equation gives

$$6x = 4.$$

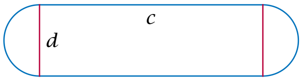
Thus $x = \frac{2}{3}$. Substituting this value of x into the equation $x^2 + y^2 = 4$ gives the equation $y^2 = \frac{32}{9}$. Thus $y = \pm\frac{\sqrt{32}}{3} = \pm\frac{4\sqrt{2}}{3}$. Hence the two points we seek are $\left(\frac{2}{3}, \frac{4\sqrt{2}}{3}\right)$ and $\left(\frac{2}{3}, -\frac{4\sqrt{2}}{3}\right)$.

39. Suppose you want to design a 400-meter track consisting of two half-circles connected by parallel line segments. Suppose also that you want the total length of the curved part of the track to equal half the total length of the straight part of the track. What dimensions should the track have?

SOLUTION Let t equal the total length of the straight part of the track in meters. We want the curved part of the track to have total length $\frac{t}{2}$. Thus we want

$$t + \frac{t}{2} = 400.$$

Solving this equation for t, we get $t = \frac{800}{3}$ meters. Thus each of the two straight pieces must be $\frac{400}{3}$ meters long. Hence we take $c = \frac{400}{3} \approx 133.33$ meters in the figure in the below.



We also want the total length of the two half-circles to be $\frac{400}{3}$ meters. Thus we want $\pi d = \frac{400}{3}$. Hence we take $d = \frac{400}{3\pi} \approx 42.44$ meters in the figure above.

41. Suppose a rope is just long enough to cover the equator of the Earth. About how much longer would the rope need to be so that it could be suspended seven feet above the entire equator?

SOLUTION Assume that the equator of the Earth is a circle. This assumption is close enough to being correct to answer a question that requires only an approximation.

Assume that the radius of the Earth is r, measured in feet (note that we do not need to know the value of r for this exercise). For a rope to cover the equator, it needs to have length $2\pi r$ feet. For a rope to be suspended seven feet above the equator, it would need to have length $2\pi(r + 7)$ feet, which equals $(2\pi r + 14\pi)$ feet. In other words, to be suspended seven feet above the equator, the rope would need to be only 14π feet longer than a rope covering the equator. Because $14\pi \approx 14 \cdot \frac{22}{7} = 44$, the rope would need to be about 44 feet longer than a rope covering the equator.

2.2 *Lines*

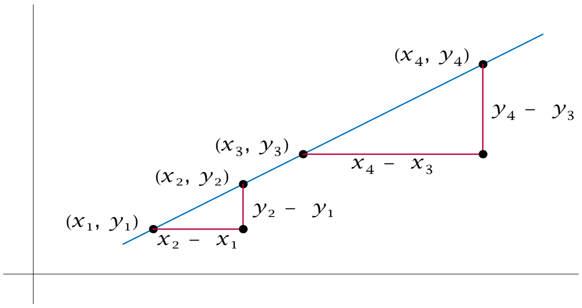
LEARNING OBJECTIVES

By the end of this section you should be able to

- find the slope of a line;
- find the equation of a line given its slope and a point on it;
- find the equation of a line given two points on it;
- determine whether or not two lines are parallel;
- find the equation of a line perpendicular to a given line and containing a given point;
- find the midpoint of a line segment.

Slope

Consider a line in the coordinate plane, along with four points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) on the line. Draw two right triangles with horizontal and vertical edges as in the figure below:



Similar triangles.

In this figure, each side of the larger triangle has twice the length of the corresponding side of the smaller triangle.

The two right triangles in the figure above are similar because their angles are equal. Thus the ratios of the corresponding sides of the two triangles above are equal. Specifically, taking the ratio of the vertical side and horizontal side for each triangle, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}.$$

The equation above states that for any pair of points (x_1, y_1) and (x_2, y_2) on the line, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ does not depend on the particular pair of points chosen on the line. If we choose another pair of points on the line, say (x_3, y_3) and (x_4, y_4) instead of (x_1, y_1) and (x_2, y_2), then the difference of second coordinates divided by the difference of first coordinates remains the same, as shown by the equation above.

Thus the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is a constant depending only on the line and not on the particular points (x_1, y_1) and (x_2, y_2) chosen on the line. This constant is called the **slope** of the line.

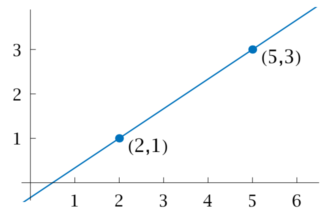
Slope

If (x_1, y_1) and (x_2, y_2) are any two points on a line, with $x_1 \neq x_2$, then the **slope** of the line is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

EXAMPLE 1

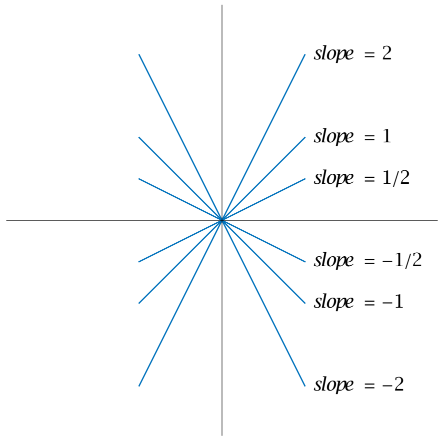
Find the slope of the line containing the points $(2, 1)$ and $(5, 3)$.



A line with slope $\frac{2}{3}$.

SOLUTION The line containing $(2, 1)$ and $(5, 3)$ is shown here. The slope of this line is $\frac{3-1}{5-2}$, which equals $\frac{2}{3}$.

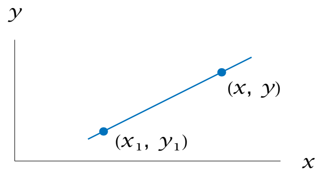
A line with positive slope slants up from left to right; a line with negative slope slants down from left to right. Lines whose slopes have larger absolute value are steeper than lines whose slopes have smaller absolute value. This figure shows some lines and their slopes; the same scale has been used on both axes.



In the figure above, the horizontal axis has slope 0, as does every horizontal line. Vertical lines, including the vertical axis, do not have a slope, because a vertical line does not contain two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$.

The Equation of a Line

Consider a line with slope m, and suppose (x_1, y_1) is a point on this line. Let (x, y) denote a typical point on the line, as shown here.



A line with slope $\frac{y - y_1}{x - x_1}$.

Because this line has slope m, we have

$$\frac{y - y_1}{x - x_1} = m.$$

Multiplying both sides of the equation above by $x - x_1$, we get the following formula:

The equation of a line, given its slope and one point on it

The line in the xy-plane that has slope m and contains the point (x_1, y_1) is given by the equation

$$y - y_1 = m(x - x_1).$$

The symbol m is often used to denote the slope of a line.

The equation above can be solved for y to get an equation for the line in the form $y = mx + b$, where m and b are constants.

EXAMPLE 2

Find the equation of the line in the xy-plane that has slope $\frac{1}{2}$ and contains $(4, 1)$.

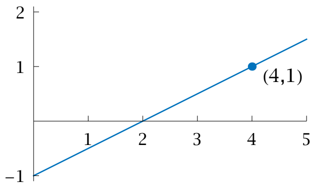
SOLUTION In this case the equation displayed above becomes

$$y - 1 = \tfrac{1}{2}(x - 4).$$

Adding 1 to both sides and simplifying, we get

$$y = \tfrac{1}{2}x - 1.$$

As a check for possible errors, if we take $x = 4$ in the equation above, we get $y = 1$. Thus the point $(4, 1)$ is indeed on this line.



The line with slope $\frac{1}{2}$ that contains the point $(4, 1)$.

As a special case of finding the equation of a line when given its slope and one point on it, suppose we want to find the equation of the line in the xy-plane with slope m that intersects the y-axis at the point $(0, b)$. In this case, the formula above becomes

$$y - b = m(x - 0).$$

*The point where a line intersects the y-axis is often called the y-**intercept**.*

Solving this equation for y, we have the following result:

The equation of a line, given its slope and y-intercept

The line in the xy-plane with slope m that intersects the y-axis at $(0, b)$ is given by the equation

$$y = mx + b.$$

EXAMPLE 3

Find the equation of the line in the xy-plane that has slope $\frac{1}{2}$ and intersects the y-axis at $(0, -1)$.

SOLUTION The formula above shows that the desired equation is

$$y = \tfrac{1}{2}x - 1.$$

This line is shown in Example 2, where you can see that the line intersects the y-axis at $(0, -1)$.

If a line contains the origin, then $b = 0$ in the formula above. For example, the line in the xy-plane that has slope 2 and contains the origin is given by the equation $y = 2x$. The figure below Example 1 shows the line $y = 2x$ and several other lines containing the origin.

Suppose now that we want to find the equation of the line containing two specific points. We can reduce this problem to a problem we have already solved by computing the slope of the line and then using the formula in the box above.

Specifically, suppose we want to find the equation of the line containing the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$. This line has slope $(y_2 - y_1)/(x_2 - x_1)$. Thus our formula for the equation of a line when given its slope and one point on it gives the following result:

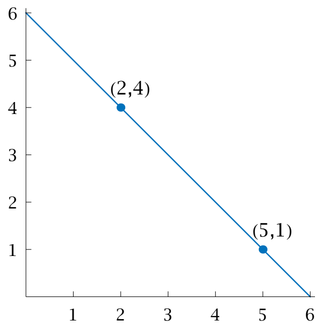
The equation of a line, given two points on it

The line in the xy-plane that contains the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is given by the equation

$$y - y_1 = \Big(\frac{y_2 - y_1}{x_2 - x_1}\Big)(x - x_1).$$

EXAMPLE 4

Find the equation of the line in the xy-plane that contains the points $(2, 4)$ and $(5, 1)$.



The line containing the points $(2, 4)$ *and* $(5, 1)$.

SOLUTION In this case the equation above becomes

$$y - 4 = \Big(\frac{1 - 4}{5 - 2}\Big)(x - 2).$$

Solving this equation for y, we get

$$y = -x + 6.$$

As a check, if we take $x = 2$ in the equation above, we get $y = 4$, and if we take $x = 5$ in the equation above, we get $y = 1$; thus the points $(2, 4)$ and $(5, 1)$ are indeed on this line.

The formula $p = 2.2k$ *is often used, but it is an approximation rather than an exact formula.*

Conversion between different units of measurement is usually done by an equation that represents one unit as a suitable multiple of another unit. For example, a pound is officially defined to be exactly 0.45359237 kilograms. Thus the equation that gives the weight k in kilograms for a object weighing p pounds is

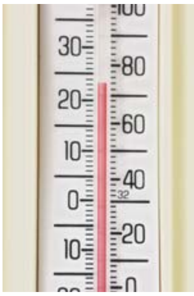
$$k = 0.45359237p.$$

Conversion between temperature scales is unusual because the zero temperature on one scale does not correspond to the zero temperature on another scale. Most other quantities such weights, lengths, and currency have the same zero point regardless of the units used. For example, without knowing the conversion rate, you know that 0 centimeters is the same length as 0 inches.

The next example shows how to find a formula for converting from Celsius temperatures to Fahrenheit temperatures. The solution to the next example also shows that rather than memorizing the formula, for example, of the equation of a line given two points on it, sometimes it is simpler just to use the available information to find the constants m and b that characterize the line $y = mx + b$.

EXAMPLE 5

Find an equation that gives the temperature F on the Fahrenheit scale corresponding to temperature C on the Celsius scale.



This thermometer shows Celsius degrees on the left, Fahrenheit degrees on the right.

SOLUTION We seek an equation of the form

$$F = mC + b$$

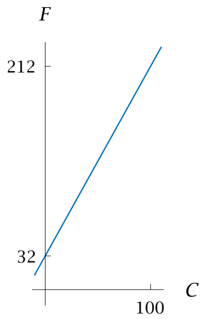
for some constants m and b.

To find m and b, we start by recalling that the freezing temperature of water equals 0 degrees Celsius and 32 degrees Fahrenheit. Plugging $C = 0$ and $F = 32$ into the equation above gives $32 = b$. Now that we know that $b = 32$, the equation above can be rewritten as

$$F = mC + 32.$$

Recall now that the boiling point of water equals 100 degrees Celsius and 212 degrees Fahrenheit. Plugging $C = 100$ and $F = 212$ into the equation above and then solving for m shows that $m = \frac{9}{5}$. Thus the formula we seek is

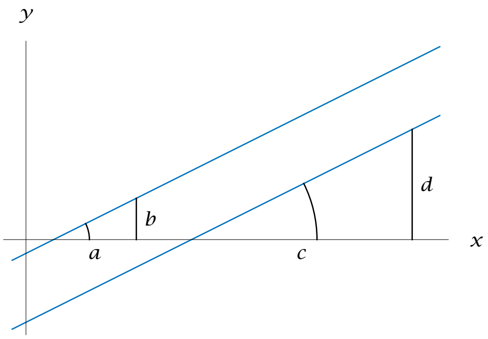
$$F = \tfrac{9}{5}C + 32.$$



The graph of $F = \frac{9}{5}C + 32$ on the interval $[-10, 110]$.

Parallel Lines

Consider two parallel lines in the coordinate plane, as shown in the figure below:



Parallel lines.

Because the two lines are parallel, the corresponding angles in the two triangles above are equal (as shown by the arcs in the figure above), and thus the two right triangles are similar. This implies that

$$\frac{b}{a} = \frac{d}{c}.$$

Because $\frac{b}{a}$ is the slope of the top line and $\frac{d}{c}$ is the slope of the bottom line, we conclude that these parallel lines have the same slope.

The logic used in the paragraph above is reversible. Specifically, suppose instead of starting with the assumption that the two lines in the figure above are parallel, we start with the assumption that the two lines have the same slope. Thus $\frac{b}{a} = \frac{d}{c}$, which implies that the two right triangles in the figure above are similar. Hence the two lines make equal angles with the horizontal axis, as shown by the arcs in the figure, which implies that the two lines are parallel.

The figure and reasoning given above do not work if both lines are horizontal or both lines are vertical. But horizontal lines all have slope 0, and the slope is not defined for vertical lines. Thus we can summarize our characterization of parallel lines as follows:

The phrase "if and only if", when connecting two statements, means that the two statements are either both true or both false. For example, $x + 1 > 6$ if and only if $x > 5$.

Parallel lines

Two nonvertical lines in the coordinate plane are parallel if and only if they have the same slope.

EXAMPLE 6

(a) Are the lines in the xy-plane given by the following equations parallel?

$$y = 4x - 5 \quad \text{and} \quad y = 4x + 18$$

(b) Are the lines in the xy-plane given by the following equations parallel?

$$y = 6x + 5 \quad \text{and} \quad y = 7x + 5$$

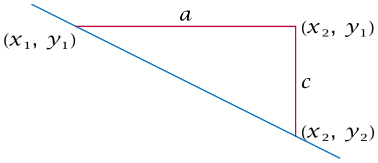
SOLUTION

(a) These lines are parallel because they have the same slope (which equals 4).

(b) These lines are not parallel because their slopes are not equal—the first line has slope 6 and the second line has slope 7.

Perpendicular Lines

Before beginning our treatment of perpendicular lines, we take a brief detour to make clear the geometry of a line with negative slope. A line with negative slope slants down from left to right. The figure below shows a line with negative slope; to avoid clutter the coordinate axes are not shown.



A line with slope $-\frac{c}{a}$.

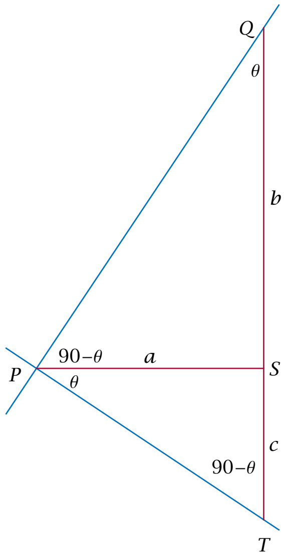
In the figure above, a is the length of the horizontal line segment and c is the length of the vertical line segment. Of course a and c are positive numbers, because lengths are positive. In terms of the coordinates as shown in the figure above, we have $a = x_2 - x_1$ and $c = y_1 - y_2$. The slope of this line equals $(y_2 - y_1)/(x_2 - x_1)$, which equals $-c/a$.

The following result gives a useful characterization of perpendicular lines:

Perpendicular lines

Two nonvertical lines are perpendicular if and only if the product of their slopes equals -1.

To explain why the result above holds, consider two perpendicular lines as shown in blue in the figure here. In addition to the two perpendicular lines in blue, the figure shows the horizontal line segment PS and the vertical line segment QT, which intersect at S.



We assume that the angle PQT is θ degrees. To check that the other three labeled angles in the figure are labeled correctly, first note that the two angles labeled $90 - \theta$ are each the third angle in a right triangle (the right triangles are PSQ and QPT), one of whose angles is θ. Consideration of the right angle QPT now shows that angle TPS is θ degrees, as labeled.

The line containing the points P and Q has slope b/a, as can be seen from the figure. Furthermore, our brief discussion of lines with negative slope shows that the line containing the points P and T has slope $-c/a$.

Consider the right triangles PSQ and TSP in the figure. These triangles have the same angles, and thus they are similar. Thus the ratios of corresponding sides are equal. Specifically, we have

$$\frac{b}{a} = \frac{a}{c}.$$

Multiplying both sides of this equation by $-c/a$, we get

$$\left(\frac{b}{a}\right) \cdot \left(-\frac{c}{a}\right) = -1.$$

As we have already seen, the first quantity on the left above is the slope of the line containing the points P and Q, and the second quantity is the slope of the line containing the points P and T. Thus we can conclude that the product of the slopes of these two perpendicular lines equals -1, as desired.

The logic used above is reversible. Specifically, suppose instead of starting with the assumption that the two lines in blue are perpendicular, we start with the assumption that the product of their slopes equals -1. This implies that $\frac{b}{a} = \frac{a}{c}$, which implies that the two right triangles PSQ and TSP are similar; thus these two triangles have the same angles. This implies that the angles are labeled correctly in the figure above (assuming that we start by declaring that angle PQS measures θ degrees). This then implies that angle QPT measures $90°$. Thus the two lines in blue are perpendicular, as desired.

Numbers m_1 and m_2 such that $m_1 m_2 = -1$ are sometimes called **negative reciprocals** *of each other.*

EXAMPLE 7

Show the lines in the xy-plane given by the following equations are perpendicular:

$$y = 4x - 5 \quad \text{and} \quad y = -\tfrac{1}{4}x + 18$$

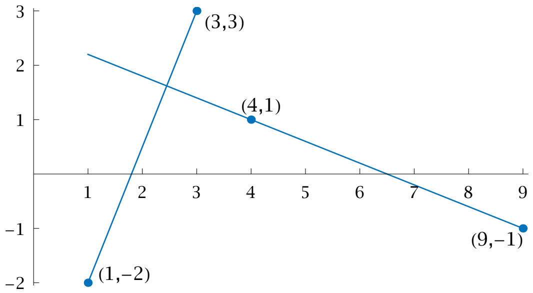
SOLUTION The first line has slope 4; the second line has slope $-\frac{1}{4}$. The product of these slopes is $4 \cdot (-\frac{1}{4})$, which equals -1. Because the product of the two slopes equals -1, the two lines are perpendicular.

To show that two lines are perpendicular, we only need to know the slopes of the lines, not their full equations, as shown by the following example.

EXAMPLE 8

Show that the line containing the points $(1, -2)$ and $(3, 3)$ is perpendicular to the line containing the points $(9, -1)$ and $(4, 1)$.

SOLUTION The line containing $(1, -2)$ and $(3, 3)$ has slope $\frac{3-(-2)}{3-1}$, which equals $\frac{5}{2}$. Also, the line containing $(9, -1)$ and $(4, 1)$ has slope $\frac{1-(-1)}{4-9}$, which equals $-\frac{2}{5}$. Because the product $\frac{5}{2} \cdot (-\frac{2}{5})$ equals -1, the two lines are perpendicular.



The line containing $(1, -2)$ and $(3, 3)$ is perpendicular to the line containing $(9, -1)$ and $(4, 1)$.

Midpoints

This subsection begins with an intuitive definition of the midpoint of a line segment:

Midpoint

The **midpoint** of a line segment is the point on the line segment that lies halfway between the two endpoints.

As you might guess, the first coordinate of the midpoint of a line segment is the average of the first coordinates of the endpoints. Similarly, the second coordinate of the midpoint is the average of the second coordinates of the endpoints. Here is the formal statement of this formula:

Midpoint

The midpoint of the line segment connecting (x_1, y_1) and (x_2, y_2) equals

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problems 51–53 at the end of this section will lead you to an explanation of why this formula for the midpoint is correct.

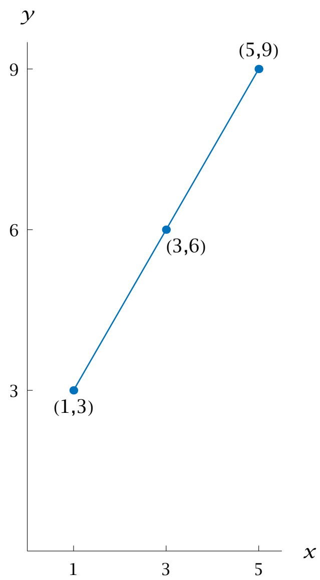
The next example illustrates the use of the formula above.

EXAMPLE 9

(a) Find the midpoint of the line segment connecting $(1, 3)$ and $(5, 9)$.

(b) Verify that the distance between the midpoint found in (a) and the first endpoint $(1, 3)$ equals the distance between the midpoint found in (a) and the second endpoint $(5, 9)$.

(c) Verify that the midpoint found in (a) lies on the line connecting $(1, 3)$ and $(5, 9)$.



The point $(3, 6)$ is the midpoint of the line segment connecting $(1, 3)$ and $(5, 9)$.

SOLUTION

(a) Using the formula above, we see that the midpoint of the line segment connecting $(1, 3)$ and $(5, 9)$ equals

$$\left(\frac{1+5}{2}, \frac{3+9}{2}\right),$$

which equals $(3, 6)$.

(b) First compute the distance between the midpoint and the endpoint $(1, 3)$:

$$\text{distance between } (3,6) \text{ and } (1,3) = \sqrt{(3-1)^2 + (6-3)^2} = \sqrt{2^2 + 3^2}$$
$$= \sqrt{13}.$$

Next compute the distance between the midpoint and the endpoint $(5, 9)$:

$$\text{distance between } (3,6) \text{ and } (5,9) = \sqrt{(3-5)^2 + (6-9)^2} = \sqrt{(-2)^2 + (-3)^2}$$
$$= \sqrt{13}.$$

As expected, these two distances are equal—the distance between the midpoint and either endpoint is $\sqrt{13}$.

(c) First compute the slope of the line containing the midpoint and the endpoint $(1, 3)$:

$$\text{slope of line containing } (3,6) \text{ and } (1,3) = \frac{6-3}{3-1} = \frac{3}{2}.$$

Next compute the slope of the line containing the midpoint and the endpoint $(5, 9)$:

$$\text{slope of line containing } (3,6) \text{ and } (5,9) = \frac{6-9}{3-5} = \frac{-3}{-2} = \frac{3}{2}.$$

As expected, these two slopes are equal. In other words, the line segment from $(1,3)$ to $(3,6)$ points in the same direction as the line segment from $(3,6)$ to $(5,9)$. Thus these three points all lie on the same line.

EXERCISES

1. What are the coordinates of the unlabeled vertex of the smaller of the two right triangles in the figure at the beginning of this section?
2. What are the coordinates of the unlabeled vertex of the larger of the two right triangles in the figure at the beginning of this section?
3. Find the slope of the line that contains the points $(3,4)$ and $(7,13)$.
4. Find the slope of the line that contains the points $(2,11)$ and $(6,-5)$.
5. Find a number t such that the line containing the points $(1,t)$ and $(3,7)$ has slope 5.
6. Find a number c such that the line containing the points $(c,4)$ and $(-2,9)$ has slope -3.
7. Suppose the tuition per semester at Euphoria State University is \$525 plus \$200 for each class unit taken.
   (a) Find an equation that gives the tuition t in dollars for taking u class units.
   (b) Find the total tuition for accumulating 120 units over 8 semesters.
   (c) Find the total tuition for accumulating 120 units over 12 semesters.
8. Suppose the tuition per semester at Luxim University is \$900 plus \$850 for each class unit taken.
   (a) Find an equation that gives the tuition t in dollars for taking u class units.
   (b) Find the total tuition for accumulating 120 units over 8 semesters.
   (c) Find the total tuition for accumulating 120 units over 10 semesters.
9. Suppose your cell phone company offers two calling plans. The pay-per-call plan charges \$14 per month plus 3 cents for each minute. The unlimited-calling plan charges a flat rate of \$29 per month for unlimited calls.
   (a) Find an equation that gives the cost c in dollars for making m minutes of phone calls per month on the pay-per-call plan.
   (b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?
10. Suppose your cell phone company offers two calling plans. The pay-per-call plan charges \$11 per month plus 4 cents for each minute. The unlimited-calling plan charges a flat rate of \$25 per month for unlimited calls.
    (a) Find an equation that gives the cost c in dollars for making m minutes of phone calls per month on the pay-per-call plan.
    (b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?
11. Find the equation of the line in the xy-plane with slope 2 that contains the point $(7,3)$.
12. Find the equation of the line in the xy-plane with slope -4 that contains the point $(-5,-2)$.
13. Find the equation of the line that contains the points $(2,-1)$ and $(4,9)$.
14. Find the equation of the line that contains the points $(-3,2)$ and $(-5,7)$.
15. Find a number t such that the point $(3,t)$ is on the line containing the points $(7,6)$ and $(14,10)$.
16. Find a number t such that the point $(-2,t)$ is on the line containing the points $(5,-2)$ and $(10,-8)$.

17. Find a formula for the number of seconds s in d days.

18. Find a formula for the number of seconds s in w weeks.

19. Find a formula for the number of inches I in M miles.

20. Find a formula for the number of miles M in F feet.

21. Find a formula for the number of kilometers k in M miles.
[*The exact conversion between the English measurement system and the metric system is given by the equation* 1 *inch* = 2.54 *centimeters.*]

22. Find a formula for the number of miles M in m meters.

23. Find a formula for the number of inches I in c centimeters.

24. Find a formula for the number of meters m in F feet.

25. Find a number c such that the point $(c, 13)$ is on the line containing the points $(-4, -17)$ and $(6, 33)$.

26. Find a number c such that the point $(c, -19)$ is on the line containing the points $(2, 1)$ and $(4, 9)$.

27. Find a number t such that the point $(t, 2t)$ is on the line containing the points $(3, -7)$ and $(5, -15)$.

28. Find a number t such that the point $(t, \frac{t}{2})$ is on the line containing the points $(2, -4)$ and $(-3, -11)$.

29. Find the equation of the line in the xy-plane that contains the point $(3, 2)$ and that is parallel to the line $y = 4x - 1$.

30. Find the equation of the line in the xy-plane that contains the point $(-4, -5)$ and that is parallel to the line $y = -2x + 3$.

31. Find the equation of the line that contains the point $(2, 3)$ and that is parallel to the line containing the points $(7, 1)$ and $(5, 6)$.

32. Find the equation of the line that contains the point $(-4, 3)$ and that is parallel to the line containing the points $(3, -7)$ and $(6, -9)$.

33. Find a number t such that the line containing the points $(t, 2)$ and $(3, 5)$ is parallel to the line containing the points $(-1, 4)$ and $(-3, -2)$.

34. Find a number t such that the line containing the points $(-3, t)$ and $(2, -4)$ is parallel to the line containing the points $(5, 6)$ and $(-2, 4)$.

35. Find the intersection in the xy-plane of the lines $y = 5x + 3$ and $y = -2x + 1$.

36. Find the intersection in the xy-plane of the lines $y = -4x + 5$ and $y = 5x - 2$.

37. Find a number b such that the three lines in the xy-plane given by the equations $y = 2x + b$, $y = 3x - 5$, and $y = -4x + 6$ have a common intersection point.

38. Find a number m such that the three lines in the xy-plane given by the equations $y = mx + 3$, $y = 4x + 1$, and $y = 5x + 7$ have a common intersection point.

39. Find the equation of the line in the xy-plane that contains the point $(4, 1)$ and that is perpendicular to the line whose equation is $y = 3x + 5$.

40. Find the equation of the line in the xy-plane that contains the point $(-3, 2)$ and that is perpendicular to the line whose equation is $y = -5x + 1$.

41. Find a number t such that the line in the xy-plane containing the points $(t, 4)$ and $(2, -1)$ is perpendicular to the line $y = 6x - 7$.

42. Find a number t such that the line in the xy-plane containing the points $(-3, t)$ and $(4, 3)$ is perpendicular to the line $y = -5x + 999$.

43. Find a number t such that the line containing the points $(4, t)$ and $(-1, 6)$ is perpendicular to the line that contains the points $(3, 5)$ and $(1, -2)$.

44. Find a number t such that the line containing the points $(t, -2)$ and $(-3, 5)$ is perpendicular to the line that contains the points $(4, 7)$ and $(1, 11)$.

45. Find the midpoint of the line segment connecting $(-3, 4)$ and $(5, 7)$.

46. Find the midpoint of the line segment connecting $(6, -5)$ and $(-3, -8)$.

47. Find numbers x and y such that $(-2, 5)$ is the midpoint of the line segment connecting $(3, 1)$ and (x, y).

48. Find numbers x and y such that $(3, -4)$ is the midpoint of the line segment connecting $(-2, 5)$ and (x, y).

PROBLEMS

49. The Kelvin temperature scale is defined by $K = C + 273.15$, where K is the temperature on the Kelvin scale and C is the temperature on the Celsius scale. (Thus -273.15 degrees Celsius, which is the temperature at which all molecular movement ceases and thus is the lowest possible temperature, corresponds to 0 on the Kelvin scale.)

    (a) Find an equation that gives the temperature F on the Fahrenheit scale corresponding to temperature K on the Kelvin scale.

    (b) Explain why the graph of the equation from part (a) is parallel to the graph of the equation obtained in Example 5.

50. Find the equation of the line in the xy-plane that has slope m and intersects the x-axis at $(c, 0)$.

51. Suppose (x_1, y_1) and (x_2, y_2) are the endpoints of a line segment.

    (a) Show that the distance between the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_1, y_1) equals half the length of the line segment.

    (b) Show that the distance between the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_2, y_2) equals half the length of the line segment.

52. Suppose (x_1, y_1) and (x_2, y_2) are the endpoints of a line segment.

    (a) Show that the line containing the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_1, y_1) has slope $\frac{y_2-y_1}{x_2-x_1}$.

    (b) Show that the line containing the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and the endpoint (x_2, y_2) has slope $\frac{y_2-y_1}{x_2-x_1}$.

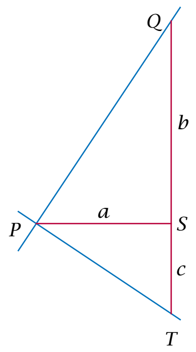
    (c) Explain why parts (a) and (b) of this problem imply that the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ lies on the line containing the endpoints (x_1, y_1) and (x_2, y_2).

53. Explain why the two previous problems imply that $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is the midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2).

54. Find six distinct points on the circle with center $(2, 3)$ and radius 5.

55. Find six distinct points on the circle with center $(4, 1)$ and circumference 3.

56. We used similar triangles to show that the product of the slopes of two perpendicular lines equals -1. The steps below outline an alternative proof that avoids the use of similar triangles but uses more algebra instead. Use the figure below, which is the same as the figure used earlier except that there is now no need to label the angles.



QP is perpendicular to PT.

    (a) Apply the Pythagorean Theorem to triangle PSQ to find the length of the line segment PQ in terms of a and b.

    (b) Apply the Pythagorean Theorem to triangle PST to find the length of the line segment PT in terms of a and c.

    (c) Apply the Pythagorean Theorem to triangle QPT to find the length of the line segment QT in terms of the lengths of the line segments of PQ and PT calculated in the first two parts of this problem.

    (d) As can be seen from the figure, the length of the line segment QT equals $b + c$. Thus set the formula for length of the line segment QT, as calculated in the previous part of this problem, equal to $b + c$, and solve the resulting equation for c in terms of a and b.

    (e) Use the result in the previous part of this problem to show that the slope of the line containing P and Q times the slope of the line containing P and T equals -1.

57. Suppose a and b are nonzero numbers. Where does the line in the xy-plane given by the equation
$$\frac{x}{a} + \frac{y}{b} = 1$$
intersect the coordinate axes?

58. Show that the points $(-84, -14)$, $(21, 1)$, and $(98, 12)$ lie on a line.

59. Show that the points $(-8, -65)$, $(1, 52)$, and $(3, 77)$ do not lie on a line.

60. Change just one of the six numbers in the problem above so that the resulting three points do lie on a line.

61. Show that for every number t, the point $(5 - 3t, 7 - 4t)$ is on the line containing the points $(2, 3)$ and $(5, 7)$.

WORKED-OUT SOLUTIONS *to Odd-numbered Exercises*

1. What are the coordinates of the unlabeled vertex of the smaller of the two right triangles in the figure at the beginning of this section?

SOLUTION Drawing vertical and horizontal lines from the point in question to the coordinate axes shows that the coordinates of the point are (x_2, y_1).

3. Find the slope of the line that contains the points $(3, 4)$ and $(7, 13)$.

SOLUTION The line containing the points $(3, 4)$ and $(7, 13)$ has slope
$$\frac{13 - 4}{7 - 3},$$
which equals $\frac{9}{4}$.

5. Find a number t such that the line containing the points $(1, t)$ and $(3, 7)$ has slope 5.

SOLUTION The slope of the line containing the points $(1, t)$ and $(3, 7)$ equals
$$\frac{7 - t}{3 - 1},$$
which equals $\frac{7-t}{2}$. We want this slope to equal 5. Thus we must find a number t such that
$$\frac{7 - t}{2} = 5.$$
Solving this equation for t, we get $t = -3$.

7. Suppose the tuition per semester at Euphoria State University is \$525 plus \$200 for each class unit taken.

(a) Find an equation that gives the tuition t in dollars for taking u class units.

(b) Find the total tuition for accumulating 120 units over 8 semesters.

(c) Find the total tuition for accumulating 120 units over 12 semesters.

SOLUTION

(a) The tuition t in dollars for taking u class units is given by the equation
$$t = 200u + 525.$$

(b) Taking 120 units over 8 semesters will cost $200 \times 120 + 8 \times 525$ dollars, which equals \$28,200.

(c) Taking 120 units over 12 semesters will cost $200 \times 120 + 12 \times 525$ dollars, which equals \$30,300.

9. Suppose your cell phone company offers two calling plans. The pay-per-call plan charges \$14 per month plus 3 cents for each minute. The unlimited-calling plan charges a flat rate of \$29 per month for unlimited calls.

(a) Find an equation that gives the cost c in dollars for making m minutes of phone calls per month on the pay-per-call plan.

(b) How many minutes per month must you use for the unlimited-calling plan to become cheaper?

SOLUTION

(a) As usual, units must be consistent throughout any calculation. Thus we think of 3 cents as 0.03 dollars. Hence the cost c in dollars for making m minutes of phone calls per month is given by the equation
$$c = 0.03m + 14.$$

(b) Setting $c = 29$ in the equation above gives the equation $29 = 0.03m + 14$. Thus

$$m = \frac{29-14}{0.03} = \frac{15}{0.03} = \frac{1500}{3} = 500.$$

Thus the two plans cost an equal amount if 500 minutes per month are used. If more than 500 minutes per month are used, then the unlimited-calling plan is cheaper.

11. Find the equation of the line in the xy-plane with slope 2 that contains the point $(7,3)$.

SOLUTION If (x,y) denotes a typical point on the line with slope 2 that contains the point $(7,3)$, then

$$\frac{y-3}{x-7} = 2.$$

Multiplying both sides of this equation by $x - 7$ and then adding 3 to both sides gives the equation

$$y = 2x - 11.$$

CHECK The line whose equation is $y = 2x - 11$ has slope 2. We should also check that the point $(7,3)$ is on this line. In other words, we need to verify the alleged equation

$$3 \stackrel{?}{=} 2 \cdot 7 - 11.$$

Simple arithmetic shows that this is indeed true.

13. Find the equation of the line that contains the points $(2,-1)$ and $(4,9)$.

SOLUTION The line that contains the points $(2,-1)$ and $(4,9)$ has slope

$$\frac{9-(-1)}{4-2},$$

which equals 5. Thus if (x,y) denotes a typical point on this line, then

$$\frac{y-9}{x-4} = 5.$$

Multiplying both sides of this equation by $x - 4$ and then adding 9 to both sides gives the equation

$$y = 5x - 11.$$

CHECK We need to check that both $(2,-1)$ and $(4,9)$ are on the line whose equation is $y = 5x - 11$. In other words, we need to verify the alleged equations

$$-1 \stackrel{?}{=} 5 \cdot 2 - 11 \quad \text{and} \quad 9 \stackrel{?}{=} 5 \cdot 4 - 11.$$

Simple arithmetic shows that both alleged equations are indeed true.

15. Find a number t such that the point $(3,t)$ is on the line containing the points $(7,6)$ and $(14,10)$.

SOLUTION First we find the equation of the line containing the points $(7,6)$ and $(14,10)$. To do this, note that the line containing those two points has slope

$$\frac{10-6}{14-7},$$

which equals $\frac{4}{7}$. Thus if (x,y) denotes a typical point on this line, then

$$\frac{y-6}{x-7} = \frac{4}{7}.$$

Multiplying both sides of this equation by $x - 7$ and then adding 6 gives the equation

$$y = \tfrac{4}{7}x + 2.$$

Now we can find a number t such that the point $(3,t)$ is on the line given by the equation above. To do this, in the equation above replace x by 3 and y by t, getting

$$t = \tfrac{4}{7} \cdot 3 + 2.$$

Performing the arithmetic to compute the right side, we get $t = \frac{26}{7}$.

CHECK We should check that all three points $(7,6)$, $(14,10)$, and $(3, \frac{26}{7})$ are on the line $y = \frac{4}{7}x + 2$. In other words, we need to verify the alleged equations

$$6 \stackrel{?}{=} \tfrac{4}{7} \cdot 7 + 2, \quad 10 \stackrel{?}{=} \tfrac{4}{7} \cdot 14 + 2, \quad \tfrac{26}{7} \stackrel{?}{=} \tfrac{4}{7} \cdot 3 + 2.$$

Simple arithmetic shows that all three alleged equations are indeed true.

17. Find a formula for the number of seconds s in d days.

SOLUTION Each minute has 60 seconds, and each hour has 60 minutes. Thus each hour has

60×60 seconds, or 3600 seconds. Each day has 24 hours; thus each day has 24×3600 seconds, or 86400 seconds. Thus

$$s = 86400d.$$

19. Find a formula for the number of inches I in M miles.

SOLUTION Each foot has 12 inches, and each mile has 5280 feet. Thus each mile has 5280×12 inches, or 63360 inches. Thus

$$I = 63360M.$$

21. Find a formula for the number of kilometers k in M miles.

SOLUTION Multiplying both sides of the equation

$$1 \text{ inch} = 2.54 \text{ centimeters}$$

by 12 gives

$$\begin{aligned} 1 \text{ foot} &= 12 \times 2.54 \text{ centimeters} \\ &= 30.48 \text{ centimeters}. \end{aligned}$$

Multiplying both sides of the equation above by 5280 gives

$$\begin{aligned} 1 \text{ mile} &= 5280 \times 30.48 \text{ centimeters} \\ &= 160934.4 \text{ centimeters} \\ &= 1609.344 \text{ meters} \\ &= 1.609344 \text{ kilometers}. \end{aligned}$$

Multiplying both sides of the equation above by a number M shows that M miles = $1.609344M$ kilometers. In other words,

$$k = 1.609344M.$$

[*The formula above is exact. However, the approximation* $k = 1.61M$ *is often used.*]

23. Find a formula for the number of inches I in c centimeters.

SOLUTION Dividing both sides of the equation

$$1 \text{ inch} = 2.54 \text{ centimeters}$$

by 2.54 gives

$$1 \text{ centimeter} = \frac{1}{2.54} \text{ inches}.$$

Multiplying both sides of the equation above by a number c shows that c centimeters = $\frac{c}{2.54}$ inches. In other words,

$$I = \frac{c}{2.54}.$$

25. Find a number c such that the point $(c, 13)$ is on the line containing the points $(-4, -17)$ and $(6, 33)$.

SOLUTION First we find the equation of the line containing the points $(-4, -17)$ and $(6, 33)$. To do this, note that the line containing those two points has slope

$$\frac{33 - (-17)}{6 - (-4)},$$

which equals 5. Thus if (x, y) denotes a typical point on this line, then

$$\frac{y - 33}{x - 6} = 5.$$

Multiplying both sides of this equation by $x - 6$ and then adding 33 gives the equation

$$y = 5x + 3.$$

Now we can find a number c such that the point $(c, 13)$ is on the line given by the equation above. To do this, in the equation above replace x by c and y by 13, getting

$$13 = 5c + 3.$$

Solving this equation for c, we get $c = 2$.

CHECK We should check that the three points $(-4, -17)$, $(6, 33)$, and $(2, 13)$ are all on the line whose equation is $y = 5x + 3$. In other words, we need to verify the alleged equations

$$-17 \stackrel{?}{=} 5 \cdot (-4) + 3, \quad 33 \stackrel{?}{=} 5 \cdot 6 + 3, \quad 13 \stackrel{?}{=} 5 \cdot 2 + 2.$$

Simple arithmetic shows that all three alleged equations are indeed true.

27. Find a number t such that the point $(t, 2t)$ is on the line containing the points $(3, -7)$ and $(5, -15)$.

SOLUTION First we find the equation of the line containing the points $(3, -7)$ and $(5, -15)$. To do this, note that the line containing those two points has slope

$$\frac{-7-(-15)}{3-5},$$

which equals -4. Thus if (x, y) denotes a point on this line, then

$$\frac{y-(-7)}{x-3} = -4.$$

Multiplying both sides of this equation by $x - 3$ and then subtracting 7 gives the equation

$$y = -4x + 5.$$

Now we can find a number t such that the point $(t, 2t)$ is on the line given by the equation above. To do this, in the equation above replace x by t and y by $2t$, getting

$$2t = -4t + 5.$$

Solving this equation for t, we get $t = \frac{5}{6}$.

CHECK We should check that the three points $(3, -7)$, $(5, -15)$, and $(\frac{5}{6}, 2 \cdot \frac{5}{6})$ are all on the line whose equation is $y = -4x + 5$. In other words, we need to verify the alleged equations

$$-7 \stackrel{?}{=} -4 \cdot 3 + 5, \quad -15 \stackrel{?}{=} -4 \cdot 5 + 5, \quad \frac{5}{3} \stackrel{?}{=} -4 \cdot \frac{5}{6} + 5.$$

Simple arithmetic shows that all three alleged equations are indeed true.

29. Find the equation of the line in the xy-plane that contains the point $(3, 2)$ and that is parallel to the line $y = 4x - 1$.

SOLUTION The line in the xy-plane whose equation is $y = 4x - 1$ has slope 4. Thus each line parallel to it also has slope 4 and hence has the form

$$y = 4x + b$$

for some constant b.

Thus we need to find a constant b such that the point $(3, 2)$ is on the line given by the equation above. Replacing x by 3 and replacing y by 2 in the equation above, we have

$$2 = 4 \cdot 3 + b.$$

Solving this equation for b, we get $b = -10$. Thus the line that we seek is described by the equation

$$y = 4x - 10.$$

31. Find the equation of the line that contains the point $(2, 3)$ and that is parallel to the line containing the points $(7, 1)$ and $(5, 6)$.

SOLUTION The line containing the points $(7, 1)$ and $(5, 6)$ has slope

$$\frac{6-1}{5-7},$$

which equals $-\frac{5}{2}$. Thus each line parallel to it also has slope $-\frac{5}{2}$ and hence has the form

$$y = -\tfrac{5}{2}x + b$$

for some constant b.

Thus we need to find a constant b such that the point $(2, 3)$ is on the line given by the equation above. Replacing x by 2 and replacing y by 3 in the equation above, we have

$$3 = -\tfrac{5}{2} \cdot 2 + b.$$

Solving this equation for b, we get $b = 8$. Thus the line that we seek is described by the equation

$$y = -\tfrac{5}{2}x + 8.$$

33. Find a number t such that the line containing the points $(t, 2)$ and $(3, 5)$ is parallel to the line containing the points $(-1, 4)$ and $(-3, -2)$.

SOLUTION The line containing the points $(-1, 4)$ and $(-3, -2)$ has slope

$$\frac{4-(-2)}{-1-(-3)},$$

which equals 3. Thus each line parallel to it also has slope 3.

The line containing the points $(t, 2)$ and $(3, 5)$ has slope

$$\frac{5-2}{3-t},$$

which equals $\frac{3}{3-t}$. From the paragraph above, we want this slope to equal 3. In other words, we need to solve the equation

$$\frac{3}{3-t} = 3.$$

Dividing both sides of the equation above by 3 and then multiplying both sides by $3 - t$ gives the equation $1 = 3 - t$. Thus $t = 2$.

35. Find the intersection in the xy-plane of the lines $y = 5x + 3$ and $y = -2x + 1$.

SOLUTION Setting the two right sides of the equations above equal to each other, we get

$$5x + 3 = -2x + 1.$$

To solve this equation for x, add $2x$ to both sides and then subtract 3 from both sides, getting $7x = -2$. Thus $x = -\frac{2}{7}$.

To find the value of y at the intersection point, we can plug the value $x = -\frac{2}{7}$ into either of the equations of the two lines. Choosing the first equation, we have $y = -5 \cdot \frac{2}{7} + 3$, which implies that $y = \frac{11}{7}$. Thus the two lines intersect at the point $(-\frac{2}{7}, \frac{11}{7})$.

CHECK As a check, we can substitute the value $x = -\frac{2}{7}$ into the equation for the second line and see if that also gives the value $y = \frac{11}{7}$. In other words, we need to verify the alleged equation

$$\tfrac{11}{7} \stackrel{?}{=} -2(-\tfrac{2}{7}) + 1.$$

Simple arithmetic shows that this is true. Thus we indeed have the correct solution.

37. Find a number b such that the three lines in the xy-plane given by the equations $y = 2x + b$, $y = 3x - 5$, and $y = -4x + 6$ have a common intersection point.

SOLUTION The unknown b appears in the first equation; thus our first step will be to find the point of intersection of the last two lines. To do this, we set the right sides of the last two equations equal to each other, getting

$$3x - 5 = -4x + 6.$$

To solve this equation for x, add $4x$ to both sides and then add 5 to both sides, getting $7x = 11$. Thus $x = \frac{11}{7}$. Substituting this value of x into the equation $y = 3x - 5$, we get

$$y = 3 \cdot \tfrac{11}{7} - 5.$$

Thus $y = -\frac{2}{7}$.

At this stage, we have shown that the lines given by the equations $y = 3x - 5$ and $y = -4x + 6$ intersect at the point $(\frac{11}{7}, -\frac{2}{7})$. We want the line given by the equation $y = 2x + b$ also to contain this point. Thus we set $x = \frac{11}{7}$ and $y = -\frac{2}{7}$ in this equation, getting

$$-\tfrac{2}{7} = 2 \cdot \tfrac{11}{7} + b.$$

Solving this equation for b, we get $b = -\frac{24}{7}$.

CHECK As a check that the line given by the equation $y = -4x + 6$ contains the point $(\frac{11}{7}, -\frac{2}{7})$, we can substitute the value $x = \frac{11}{7}$ into the equation for that line and see if it gives the value $y = -\frac{2}{7}$. In other words, we need to verify the alleged equation

$$-\tfrac{2}{7} \stackrel{?}{=} -4 \cdot \tfrac{11}{7} + 6.$$

Simple arithmetic shows that this is true. Thus we indeed found the correct point of intersection.

We chose the line whose equation is given by $y = -4x + 6$ for this check because the other two lines had been used in direct calculations in our solution.

39. Find the equation of the line in the xy-plane that contains the point $(4, 1)$ and that is perpendicular to the line whose equation is $y = 3x + 5$.

SOLUTION The line in the xy-plane whose equation is $y = 3x + 5$ has slope 3. Thus every line perpendicular to it has slope $-\frac{1}{3}$. Hence the equation of the line that we seek has the form

$$y = -\tfrac{1}{3}x + b$$

for some constant b. We want the point $(4, 1)$ to be on this line. Substituting $x = 4$ and $y = 1$ into the equation above, we have

$$1 = -\tfrac{1}{3} \cdot 4 + b.$$

Solving this equation for b, we get $b = \frac{7}{3}$. Thus the equation of the line that we seek is

$$y = -\tfrac{1}{3}x + \tfrac{7}{3}.$$

41. Find a number t such that the line in the xy-plane containing the points $(t, 4)$ and $(2, -1)$ is perpendicular to the line $y = 6x - 7$.

SOLUTION The line in the xy-plane whose equation is $y = 6x - 7$ has slope 6. Thus every line perpendicular to it has slope $-\frac{1}{6}$. Thus we want the line containing the points $(t, 4)$ and $(2, -1)$ to have slope $-\frac{1}{6}$. In other words, we want

$$\frac{4 - (-1)}{t - 2} = -\frac{1}{6}.$$

Solving this equation for t, we get $t = -28$.

43. Find a number t such that the line containing the points $(4, t)$ and $(-1, 6)$ is perpendicular to the line that contains the points $(3, 5)$ and $(1, -2)$.

SOLUTION The line containing the points $(3, 5)$ and $(1, -2)$ has slope

$$\frac{5 - (-2)}{3 - 1},$$

which equals $\frac{7}{2}$. Thus every line perpendicular to it has slope $-\frac{2}{7}$. Thus we want the line containing the points $(4, t)$ and $(-1, 6)$ to have slope $-\frac{2}{7}$. In other words, we want

$$\frac{t - 6}{4 - (-1)} = -\frac{2}{7}.$$

Solving this equation for t, we get $t = \frac{32}{7}$.

45. Find the midpoint of the line segment connecting $(-3, 4)$ and $(5, 7)$.

SOLUTION The midpoint of the line segment connecting $(-3, 4)$ and $(5, 7)$ is

$$\left(\frac{-3 + 5}{2}, \frac{4 + 7}{2}\right),$$

which equals $(1, \frac{11}{2})$.

47. Find numbers x and y such that $(-2, 5)$ is the midpoint of the line segment connecting $(3, 1)$ and (x, y).

SOLUTION The midpoint of the line segment connecting $(3, 1)$ and (x, y) is

$$\left(\frac{3 + x}{2}, \frac{1 + y}{2}\right).$$

We want this to equal $(-2, 5)$. Thus we must solve the equations

$$\frac{3 + x}{2} = -2 \quad \text{and} \quad \frac{1 + y}{2} = 5.$$

Solving these equations gives $x = -7$ and $y = 9$.

2.3 *Quadratic Expressions and Conic Sections*

LEARNING OBJECTIVES

By the end of this section you should be able to

- rewrite a quadratic expression using the completing-the-square technique;
- find the point on a line closest to a given point;
- derive and use the quadratic formula;
- find the center and radius of a circle from its equation;
- find the vertex of a parabola;
- recognize and work with equations of ellipses and hyperbolas.

Completing the Square

Suppose you want to find the numbers x such that

$$x^2 - 5 = 0.$$

The solution is easy to obtain. Simply add 5 to both sides of the equation above, getting $x^2 = 5$, and then conclude that $x = \pm\sqrt{5}$.

Here $\pm$ indicates that we can choose either the plus sign or the minus sign.

Now consider the harder problem of finding the numbers x such that

$$x^2 + 6x - 4 = 0.$$

Nothing obvious works here. Adding either 4 or $4 - 6x$ to both sides of this equation does not produce a new equation that leads to the solution.

However, a technique called **completing the square** can be used to deal with equations such as the one above. The key to this technique is the identity

$$(x + t)^2 = x^2 + 2tx + t^2.$$

Be sure that you are thoroughly familiar with this crucial identity.

The next example illustrates the technique of completing the square.

EXAMPLE 1

Find the numbers x such that

$$x^2 + 6x - 4 = 0.$$

SOLUTION The idea here is that we want the $6x$ term in the equation above to match term $2tx$ term in the expansion above for $(x + t)^2$. In other words, we want $2t = 6$, and hence we choose $t = 3$.

When $(x + 3)^2$ is expanded, we get $x^2 + 6x + 9$. The x^2 and $6x$ terms match the corresponding terms in the expression $x^2 + 6x$, but the expansion of $(x + 3)^2$ has an extra constant term of 9. Thus we subtract 9, rewriting $x^2 + 6x$ in the equation above as $(x + 3)^2 - 9$, getting

$$(x + 3)^2 - 9 - 4 = 0.$$

Now add 13 to both sides of the equation above, getting $(x+3)^2 = 13$, which implies that $x + 3 = \pm\sqrt{13}$. Finally, add -3 to both sides, concluding that $x = -3 \pm \sqrt{13}$.

The general formula for the substitution to make when completing the square is shown below. Do not memorize this formula. You only need to remember that the coefficient of the x term will need to be divided by 2, and then the appropriate constant will need to be subtracted to get a correct identity.

Completing the square

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

For example, if $b = -10$, then the identity above becomes

$$x^2 - 10x = (x-5)^2 - 25.$$

Note that the term that is subtracted is always positive because $(\frac{b}{2})^2$ is positive regardless of whether b is positive or negative.

The next example shows the usefulness of completing the square.

EXAMPLE 2

(a) What value of x makes $3x^2 - 5x + 4$ as small as possible?

(b) What is the smallest value of $3x^2 - 5x + 4$?

SOLUTION

(a) We factor out the coefficient 3 from the x^2 and x terms and then apply the completing-the-square identity, as follows:

The new expression has fractions making it look more cumbersome than the original expression. However, this new expression allows us to answer the questions above.

$$\begin{aligned}3x^2 - 5x + 4 &= 3\left[x^2 - \tfrac{5}{3}x\right] + 4\\ &= 3\left[\left(x - \tfrac{5}{6}\right)^2 - \tfrac{25}{36}\right] + 4\\ &= 3\left(x - \tfrac{5}{6}\right)^2 - \tfrac{25}{12} + 4\\ &= 3\left(x - \tfrac{5}{6}\right)^2 + \tfrac{23}{12}\end{aligned}$$

The term $(x - \frac{5}{6})^2$ in the last expression is positive for all values of x except for $x = \frac{5}{6}$, which makes $(x - \frac{5}{6})^2$ equal to 0. Thus $3x^2 - 5x + 4$ is as small as possible when $x = \frac{5}{6}$.

(b) We could substitute the value $x = \frac{5}{6}$ into the expression $3x^2 - 5x + 4$ to find the smallest value of this expression. However, without doing any work we can see from the last equation above that this expression equals $\frac{23}{12}$ when $x = \frac{5}{6}$. Thus $\frac{23}{12}$ is the smallest value of $3x^2 - 5x + 4$.

The next example illustrates another application of the completing-the-square technique.

EXAMPLE 3

Find the point on the line $y = 2x - 1$ that is closest to the point $(2, 1)$.

SOLUTION A typical point on the line $y = 2x - 1$ has coordinates $(x, 2x - 1)$. The distance between this point and $(2, 1)$ equals

$$\sqrt{(x-2)^2 + (2x-1-1)^2},$$

which with a bit of algebra (do it!) can be rewritten as

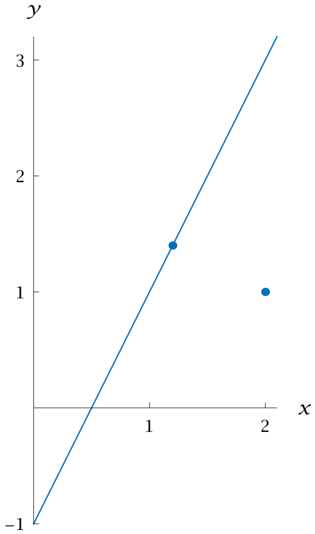
$$\sqrt{5x^2 - 12x + 8}.$$

We want to make the quantity above as small as possible, which means that we need to make $5x^2 - 12x$ as small as possible. This can be done by completing the square:

$$\begin{aligned} 5x^2 - 12x &= 5[x^2 - \tfrac{12}{5}x] \\ &= 5[(x - \tfrac{6}{5})^2 - \tfrac{36}{25}]. \end{aligned}$$

The last quantity will be as small as possible when $x = \frac{6}{5}$. Plugging $x = \frac{6}{5}$ into the equation $y = 2x - 1$ gives $y = \frac{7}{5}$.

Thus $(\frac{6}{5}, \frac{7}{5})$ is the point on the line $y = 2x - 1$ that is closest to the point $(2, 1)$. The picture in the margin shows that $(\frac{6}{5}, \frac{7}{5})$ is indeed a plausible solution.



The line $y = 2x - 1$, the point $(2, 1)$, and the point on the line closest to $(2, 1)$.

The Quadratic Formula

Quadratic expression

A **quadratic expression** has the form

$$ax^2 + bx + c,$$

where $a \neq 0$.

Consider the quadratic expression $ax^2 + bx + c$, where $a \neq 0$. Factor out a from the first two terms and then complete the square, as follows:

We will follow the same pattern as in previous concrete examples and complete the square with an arbitrary quadratic expression. This will allow us to derive the quadratic formula.

$$\begin{aligned} ax^2 + bx + c &= a\Big[x^2 + \frac{b}{a}x\Big] + c \\ &= a\Big[\Big(x + \frac{b}{2a}\Big)^2 - \frac{b^2}{4a^2}\Big] + c \\ &= a\Big(x + \frac{b}{2a}\Big)^2 - \frac{b^2}{4a} + c \\ &= a\Big(x + \frac{b}{2a}\Big)^2 - \frac{b^2 - 4ac}{4a}. \end{aligned}$$

Suppose now that we want to find the numbers x, such that $ax^2+bx+c=0$. Setting the last expression equal to 0, we get

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a},$$

and then dividing both sides by a gives

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Regardless of the value of x, the left side of the last equation is a positive number or 0. Thus if the right side is negative, the equation does not hold for any real number x. In other words, if $b^2-4ac<0$, then the equation $ax^2+bx+c=0$ has no real solutions.

If $b^2-4ac\geq 0$, then we can take the square root of both sides of the last equation, getting

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a},$$

and then adding $-\frac{b}{2a}$ to both sides gives

By completing the square, we have derived the quadratic formula!

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Quadratic formula

Consider the equation

$$ax^2+bx+c=0,$$

where a, b, and c are real numbers with $a\neq 0$.

- If $b^2-4ac<0$, then the equation above has no (real) solutions.
- If $b^2-4ac=0$, then the equation above has one solution:

$$x=-\frac{b}{2a}.$$

- If $b^2-4ac>0$, then the equation above has two solutions:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The quadratic formula often is useful in problems that do not initially seem to involve quadratic expressions. The following example illustrates how a simply stated problem can lead to a quadratic expression.

EXAMPLE 4

Find two numbers whose sum equals 7 and whose product equals 8.

SOLUTION Let's call the two numbers s and t. We want

$$s + t = 7 \quad \text{and} \quad st = 8.$$

Solving the first equation for s, we have $s = 7 - t$. Substituting this expression for s into the second equation gives $(7 - t)t = 8$, which is equivalent to the equation

$$t^2 - 7t + 8 = 0.$$

Using the quadratic formula to solve this equation for t gives

$$t = \frac{7 \pm \sqrt{7^2 - 4 \cdot 8}}{2} = \frac{7 \pm \sqrt{17}}{2}.$$

Let's choose the solution $t = \frac{7+\sqrt{17}}{2}$. Plugging this value of t into the equation $s = 7 - t$ then gives $s = \frac{7-\sqrt{17}}{2}$.

You should verify that if we had chosen $t = \frac{7-\sqrt{17}}{2}$, then we would have ended up with the same pair of numbers.

Thus two numbers whose sum equals 7 and whose product equals 8 are $\frac{7-\sqrt{17}}{2}$ and $\frac{7+\sqrt{17}}{2}$.

REMARK To check that this solution is correct, note that

$$\frac{7 - \sqrt{17}}{2} + \frac{7 + \sqrt{17}}{2} = \frac{14}{2} = 7$$

and

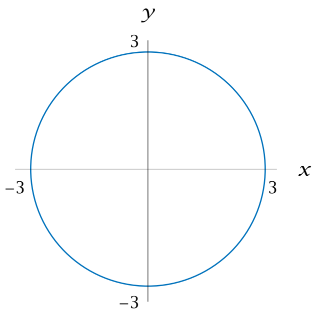
$$\frac{7 - \sqrt{17}}{2} \cdot \frac{7 + \sqrt{17}}{2} = \frac{7^2 - \sqrt{17}^2}{4} = \frac{49 - 17}{4} = \frac{32}{4} = 8.$$

Circles

The set of points that have distance 3 from the origin in a coordinate plane is the circle with radius 3 centered at the origin. The next example shows how to find an equation that describes this circle.

EXAMPLE 5

Find an equation that describes the circle with radius 3 centered at the origin in the xy-plane.



The circle of radius 3 centered at the origin.

SOLUTION Recall that the distance from a point (x, y) to the origin is $\sqrt{x^2 + y^2}$. Hence a point (x, y) has distance 3 from the origin if and only if

$$\sqrt{x^2 + y^2} = 3.$$

Squaring both sides, we get

$$x^2 + y^2 = 9.$$

More generally, suppose r is a positive number. Using the same reasoning as above, we see that

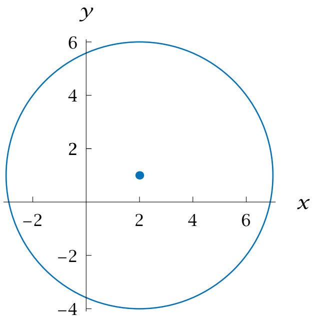
$$x^2 + y^2 = r^2$$

is the equation of the circle with radius r centered at the origin in the xy-plane.

We can also consider circles centered at points other than the origin.

EXAMPLE 6

Find the equation of the circle in the xy-plane centered at $(2, 1)$ with radius 5.



The circle centered at $(2, 1)$ with radius 5.

SOLUTION This circle is the set of points whose distance from $(2, 1)$ equals 5. In other words, the circle centered at $(2, 1)$ with radius 5 is the set of points (x, y) satisfying the equation

$$\sqrt{(x-2)^2 + (y-1)^2} = 5.$$

Squaring both sides, we can more conveniently describe this circle as the set of points (x, y) such that

$$(x-2)^2 + (y-1)^2 = 25.$$

Using the same reasoning as in the example above, we get the following more general result:

Equation of a circle

The circle with center (h, k) and radius r is the set of points (x, y) satisfying the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

For example, the equation $(x-3)^2 + (y+5)^2 = 7$ describes the circle in the xy-plane with radius $\sqrt{7}$ centered at $(3, -5)$.

Sometimes the equation of a circle may be in a form in which the radius and center are not obvious. You may then need to complete the square to find the radius and center. The following example illustrates this procedure:

EXAMPLE 7

Find the radius and center of the circle in the xy-plane described by

$$x^2 + 4x + y^2 - 6y = 12.$$

SOLUTION Completing the square, we have

$$\begin{aligned} 12 &= x^2 + 4x + y^2 - 6y \\ &= (x+2)^2 - 4 + (y-3)^2 - 9 \\ &= (x+2)^2 + (y-3)^2 - 13. \end{aligned}$$

Here the completing-the-square technique has been applied separately to the x and y variables.

Adding 13 to the first and last sides of the equation above shows that

$$(x+2)^2 + (y-3)^2 = 25.$$

Thus we have a circle with radius 5 centered at $(-2, 3)$.

Ellipses

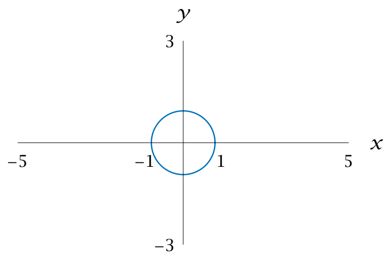
Ellipse

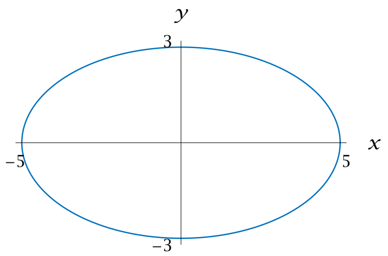
Stretching a circle horizontally and/or vertically produces a curve called an **ellipse.**

EXAMPLE 8

Find an equation describing the ellipse in the xy-plane produced by stretching a circle of radius 1 centered at the origin horizontally by a factor of 5 and vertically by a factor of 3.

SOLUTION To find an equation describing this ellipse, consider a typical point (u, v) on the circle of radius 1 centered at the origin. Thus $u^2 + v^2 = 1$.





Stretching horizontally by a factor of 5 *and vertically by a factor of* 3 *transforms the circle on the left into the ellipse on the right.*

Stretching horizontally by a factor of 5 and stretching vertically by a factor of 3 transforms the point (u, v) to the point $(5u, 3v)$. Rewrite the equation $u^2 + v^2 = 1$ in terms of this new point, getting

$$\frac{(5u)^2}{25} + \frac{(3v)^2}{9} = 1.$$

Write the transformed point $(5u, 3v)$ as (x, y), thus setting $x = 5u$ and $y = 3v$, getting

$$\frac{x^2}{25} + \frac{y^2}{9} = 1,$$

which is the equation of the ellipse shown above on the right.

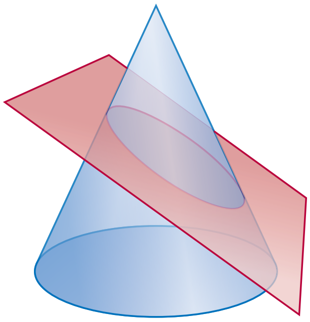
The points $(\pm 5, 0)$ and $(0, \pm 3)$ satisfy this equation and thus lie on the ellipse, as can also be seen from the figure above.

The German mathematician Johannes Kepler, who in 1609 published his discovery that the orbits of the planets are ellipses, not circles or combinations of circles as had been previously thought.

More generally, suppose a and b are positive numbers. Suppose the circle of radius 1 centered at the origin is stretched horizontally by a factor of a and stretched vertically by a factor of b. Using the same reasoning as above (just replace 5 by a and replace 3 by b), we see that the equation of the resulting ellipse in the xy-plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Viewing an ellipse as a stretched circle will lead us to the formula for the area inside an ellipse in Section 2.4.



The ancient Greeks discovered that the intersection of a cone and an appropriately positioned plane is an ellipse.

The points $(\pm a, 0)$ and $(0, \pm b)$ satisfy this equation and lie on the ellipse.

Planets have orbits that are ellipses, but you may be surprised to learn that the sun is *not* located at the center of those orbits. Instead, the sun is located at what is called a **focus** of each elliptical orbit. The plural of focus is **foci**, which are defined as follows:

Foci of an ellipse

The **foci** of an ellipse are two points with the property that the sum of the distances from the foci to a point on the ellipse is a constant independent of the point on the ellipse.

As we will see in the next example, the foci for the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ are the points $(-4, 0)$ and $(4, 0)$. This example shows how to verify that a pair of points are foci for an ellipse.

EXAMPLE 9

(a) Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ to the point $(4, 0)$.

(b) Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ to the point $(-4, 0)$.

(c) Show that $(4, 0)$ and $(-4, 0)$ are foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Isaac Newton showed that the equations of gravity imply that a planetary orbit around a star is an ellipse with the star at one of the foci. For example, if units are chosen so that the orbit of a planet is the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, then the star must be located at either $(4, 0)$ or $(-4, 0)$.

SOLUTION

(a) The distance from (x, y) to the point $(4, 0)$ is $\sqrt{(x-4)^2 + y^2}$, which equals

$$\sqrt{x^2 - 8x + 16 + y^2}.$$

We want an answer solely in terms of x, assuming that (x, y) lies on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Solving the ellipse equation for y^2, we have

$$y^2 = 9 - \tfrac{9}{25}x^2.$$

Substituting this expression for y^2 into the expression above shows that the distance from (x, y) to $(4, 0)$ equals

$$\sqrt{25 - 8x + \tfrac{16}{25}x^2},$$

which equals $\sqrt{\left(5 - \frac{4}{5}x\right)^2}$, which equals $5 - \frac{4}{5}x$.

(b) The distance from (x, y) to the point $(-4, 0)$ is $\sqrt{(x+4)^2 + y^2}$, which equals

$$\sqrt{x^2 + 8x + 16 + y^2}.$$

Now proceed as in the solution to part (a), with $-8x$ replaced by $8x$, concluding that the distance from (x, y) to $(-4, 0)$ is $5 + \frac{4}{5}x$.

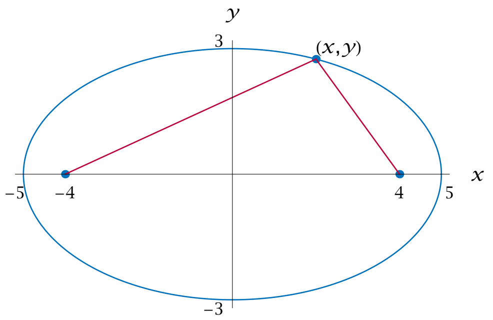
(c) As we know from the previous two parts of this example, if (x, y) is a point on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, then

$$\text{distance from } (x, y) \text{ to } (4, 0) \text{ is } 5 - \tfrac{4}{5}x$$

and

$$\text{distance from } (x, y) \text{ to } (-4, 0) \text{ is } 5 + \tfrac{4}{5}x.$$

Adding these distances, we see that the sum equals 10, which is a constant independent of the point (x, y) on the ellipse. Thus $(4, 0)$ and $(-4, 0)$ are foci of this ellipse.



For every point (x, y) on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, the sum of the lengths of the two red line segments equals 10.

No pair of points other than $(4, 0)$ and $(-4, 0)$ has the property that the sum of the distances to points on this ellipse is constant.

The next result generalizes the example above. The verification of this result is outlined in Problems 72–77. To do this verification, simply use the ideas from the example above.

Formula for the foci of an ellipse

- If $a > b > 0$, then the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are the points

$$(\sqrt{a^2 - b^2}, 0) \quad \text{and} \quad (-\sqrt{a^2 - b^2}, 0).$$

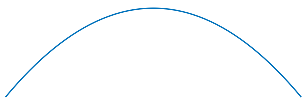
- If $b > a > 0$, then the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are the points

$$(0, \sqrt{b^2 - a^2}) \quad \text{and} \quad (0, -\sqrt{b^2 - a^2}).$$

This result does not deal with the easy case $a = b$, when the ellipse is a circle centered at the origin. In this case, the foci can be thought of as two copies of the origin.

Parabolas

A kicked or thrown football follows a path shaped like the curve in the margin. You may be tempted to think that this curve is half of an ellipse. However, the equations of gravity show that objects thrown into the air follow a path that is part of a parabola, not part of an ellipse. Thus we now turn our attention to parabolas and the equations that define them.



The path of a football.

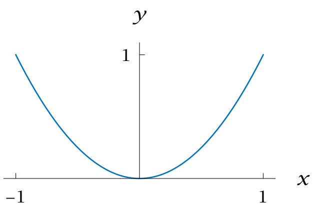
Parabolas can be defined geometrically, but for our purposes it is simpler to look at a class of parabolas that are easy to define algebraically. For now, we will restrict our attention to parabolas in the xy-plane that have a vertical line of symmetry:

Parabolas

The graph of an equation of the form

$$y = ax^2 + bx + c,$$

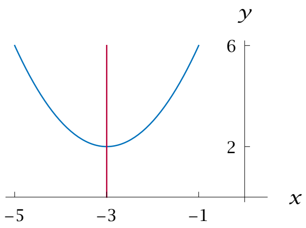
where $a \neq 0$, is called a **parabola**.



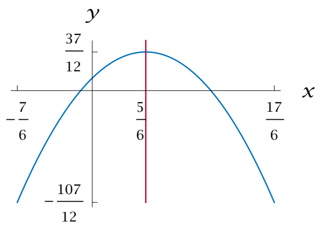
The graph of $y = x^2$ for $-1 \le x \le 1$.

For example, the graph of the equation $y = x^2$ is the familiar parabola shown in the margin. This parabola is symmetric about the y-axis, meaning that the parabola is unchanged if it is flipped across the y-axis. Note that this line of symmetry intersects this parabola at the origin, which is the lowest point on this parabola.

So that you can become familiar with the shape of parabolas, the figure below shows two more typical parabolas.



$y = x^2 + 6x + 11$
for $-5 \le x \le -1$



$y = -3x^2 + 5x + 1$
for $-\frac{7}{6} \le x \le \frac{17}{6}$

The parabola on the left is symmetric about the line $x = -3$ (shown in red), which intersects the parabola at its lowest point. The parabola on the right is symmetric about the line $x = \frac{5}{6}$ (shown in red), which intersects the parabola at its highest point.

Every parabola is symmetric about some line. The point where this line of symmetry intersects the parabola is sufficiently important to deserve its own name:

Vertex

The **vertex** of a parabola is the point where the line of symmetry of the parabola intersects the parabola.

For example, the figure above shows that the vertex of the parabola $y = x^2 + 6x + 11$ on the left is $(-3, 2)$, which is the lowest point on the graph. The figure above also shows that the vertex of the parabola $y = -3x^2 + 5x + 1$ is $(\frac{5}{6}, \frac{37}{12})$, which is the highest point on the graph.

The parabolas above exhibit the typical behavior described by the following result:

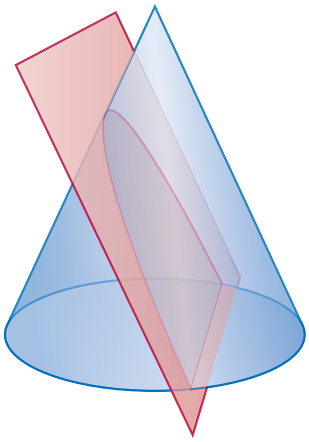
Parabola direction

Consider the parabola given by the equation

$$y = ax^2 + bx + c,$$

where $a \neq 0$.

- If $a > 0$, then the parabola opens upward and the vertex of the parabola is the lowest point on the graph.
- If $a < 0$, then the parabola opens downward and the vertex of the parabola is the highest point on the graph.



The ancient Greeks discovered that the intersection of a cone and an appropriately positioned plane is a parabola.

When we deal with function transformations in Section 3.2, we will see why the result above holds and we will learn how to use function transformations to obtain the graph of a parabola by appropriate transformations of the graph of $y = x^2$. Thus for now we give only one example showing how to find the vertex of a parabola.

EXAMPLE 10

(a) For what value of x does $x^2 + 6x + 11$ attain its minimum value?

(b) Find the vertex of the graph of $y = x^2 + 6x + 11$.

SOLUTION

(a) First complete the square to rewrite $x^2 + 6x + 11$, as follows:

$$\begin{aligned} x^2 + 6x + 11 &= (x+3)^2 - 9 + 11 \\ &= (x+3)^2 + 2 \end{aligned}$$

The last expression shows that $x^2 + 6x + 11$ takes on its minimum value when $x = -3$, because $(x+3)^2$ is positive for all values of x except $x = -3$.

(b) The last expression above shows that $x^2 + 6x + 11$ equals 2 when $x = -3$. Thus the vertex of the graph of $y = x^2 + 6x + 11$ is the point $(-3, 2)$, as shown in the graph of this parabola above.

Hyperbolas

A comet whose orbit lies on a hyperbola will come near earth at most once. A comet whose orbit is an ellipse will return periodically.

Some comets and all planets travel in orbits that are ellipses. However, many comets have orbits that are not ellipses but instead lie on hyperbolas, which is another category of curves. Thus we now turn our attention to hyperbolas and the equations that define them.

Hyperbolas can be defined geometrically, but we will restrict attention to hyperbolas in the xy-plane that can be defined in the following simple fashion:

Hyperbolas

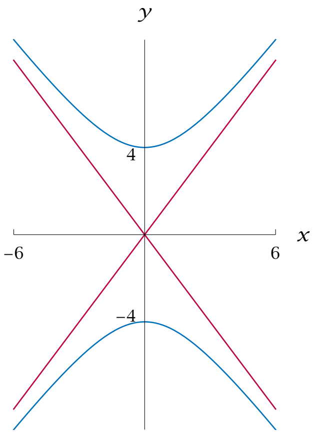
The graph of an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = c,$$

where a, b, and c are nonzero numbers, is called a **hyperbola**.

The figure below shows the graph of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ in blue for $-6 \le x \le 6$. Note that this hyperbola consists of two branches rather than the single curve we get for ellipses and parabolas. Some key properties of this graph, which are typical of hyperbolas, are discussed in the next example.

EXAMPLE 11



The hyperbola

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

in blue for $-6 \le x \le 6$, along with the lines $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$ in red.

Consider the hyperbola

$$\frac{y^2}{16} - \frac{x^2}{9} = 1.$$

(a) Explain why this hyperbola intersects the y-axis at $(0, 4)$ and $(0, -4)$.

(b) Explain why the hyperbola contains no points (x, y) with $|y| < 4$.

(c) Explain why points (x, y) on the hyperbola for large values of x are near the line $y = \frac{4}{3}x$ or the line $y = -\frac{4}{3}x$.

SOLUTION

(a) If $x = 0$, then the equation defining the hyperbola shows that $\frac{y^2}{16} = 1$. Thus $y = \pm 4$. Hence the hyperbola intersects the y-axis, which is defined by $x = 0$, at $(0, 4)$ and $(0, -4)$.

(b) If (x, y) is a point on the hyperbola, then

$$y^2 = 16\left(1 + \frac{x^2}{9}\right) \ge 16,$$

which implies that $|y| \ge 4$. Thus the hyperbola contains no points (x, y) with $|y| < 4$.

(c) The equation defining this parabola can be rewritten in the form

$$\frac{y^2}{x^2} = \frac{16}{9} + \frac{16}{x^2}$$

if $x \ne 0$. If x is large, then the equation above implies that

$$\frac{y^2}{x^2} \approx \frac{16}{9}.$$

Taking square roots then shows that $y \approx \pm\frac{4}{3}x$.

Each branch of a hyperbola may appear to be shaped like a parabola, but these curves are not parabolas. As one notable difference, a parabola cannot have the behavior described in part (c) of the example above.

Compare the following definition of the foci of a hyperbola with the definition earlier in this section of the foci of an ellipse.

Foci of a hyperbola

The **foci** of a hyperbola are two points with the property that the difference of the distances from the foci to a point on the hyperbola is a constant independent of the point on the hyperbola.

As we will see in the next example, the foci for the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ are the points $(0, -5)$ and $(0, 5)$. This example shows how to verify that a pair of points are foci for a hyperbola.

Problems 81–84 show why the graph of $y = \frac{1}{x}$ is also called a hyperbola.

EXAMPLE 12

(a) Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ to the point $(0, -5)$.

(b) Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ to the point $(0, 5)$.

(c) Show that $(0, -5)$ and $(0, 5)$ are foci of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$.

SOLUTION

(a) The distance from (x, y) to the point $(0, -5)$ is $\sqrt{x^2 + (y + 5)^2}$, which equals

$$\sqrt{x^2 + y^2 + 10y + 25}.$$

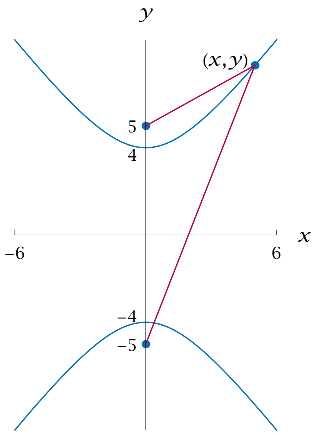
We want an answer solely in terms of y, assuming that (x, y) lies on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ and $y > 0$. Solving the hyperbola equation for x^2, we have

$$x^2 = \tfrac{9}{16}y^2 - 9.$$

Substituting this expression for x^2 into the expression above shows that the distance from (x, y) to $(0, -5)$ equals

$$\sqrt{\tfrac{25}{16}y^2 + 10y + 16},$$

which equals $\sqrt{\left(\frac{5}{4}y + 4\right)^2}$, which equals $\frac{5}{4}y + 4$.



For every point (x, y) on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$, the difference of the lengths of the two red line segments equals 8.

(b) The distance from (x, y) to the point $(0, 5)$ is $\sqrt{x^2 + (y - 5)^2}$, which equals

$$\sqrt{x^2 + y^2 - 10y + 25}.$$

Now proceed as in the solution to part (a), with $10y$ replaced by $-10y$, concluding that the distance from (x, y) to $(0, 5)$ is $\frac{5}{4}y - 4$.

(c) As we know from the previous two parts of this example, if (x, y) is a point on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ and $y > 0$, then

$$\text{distance from } (x, y) \text{ to } (0, -5) \text{ is } \tfrac{5}{4}y + 4$$

and

$$\text{distance from } (x, y) \text{ to } (0, 5) \text{ is } \tfrac{5}{4}y - 4.$$

Subtracting these distances, we see that the difference equals 8, which is a constant independent of the point (x, y) on the hyperbola. Thus $(0, -5)$ and $(0, 5)$ are foci of this hyperbola.

No pair of points other than $(0, -5)$ and $(0, 5)$ has the property that the difference of the distances to points on this hyperbola is constant.

Isaac Newton showed that a comet's orbit around a star lies either on an ellipse or on a parabola (rare) or on a hyperbola with the star at one of the foci. For example, if units are chosen so that the orbit of a comet is the upper branch of the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$, *then the star must be located at* $(0, 5)$.

The next result generalizes the example above. The verification of this result is outlined in Problems 78–80. To do this verification, simply use the ideas from the example above.

Formula for the foci of a hyperbola

If a and b are nonzero numbers, then the foci of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are the points

$$(0, -\sqrt{a^2 + b^2}) \quad \text{and} \quad (0, \sqrt{a^2 + b^2}).$$

Sometimes when a new comet is discovered, there are not enough observations to determine whether the comet is in an elliptical orbit or in a hyperbolic orbit. The distinction is important, because a comet in a hyperbolic orbit will disappear and never again be visible from earth.

EXERCISES

For Exercises 1–12, use the following information: If an object is thrown straight up into the air from height H ***feet at time*** $\mathbf{0}$ ***with initial velocity*** V ***feet per second, then at time*** t ***seconds the height of the object is***

$$-16.1t^2 + Vt + H$$

feet. This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

Some notebook computers have a sensor that detects sudden changes in motion and stops the notebook's hard drive, protecting it from damage.

1. Suppose a notebook computer is accidentally knocked off a shelf that is six feet high. How long before the computer hits the ground?

2. Suppose a notebook computer is accidentally knocked off a desk that is three feet high. How long before the computer hits the ground?

3. Suppose the motion detection/protection mechanism of a notebook computer takes 0.3 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

4. Suppose the motion detection/protection mechanism of a notebook computer takes 0.4 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

5. Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.
   (a) How long before the ball hits the ground?
   (b) How long before the ball reaches its maximum height?
   (c) What is the ball's maximum height?

6. Suppose a ball is tossed straight up into the air from height 4 feet with initial velocity 40 feet per second.
   (a) How long before the ball hits the ground?
   (b) How long before the ball reaches its maximum height?
   (c) What is the ball's maximum height?

7. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?

8. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball stay in the air for 3 seconds?

9. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach its maximum height after 1 second?

10. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach its maximum height after 2 seconds?

11. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach a height of 50 feet?

12. Suppose a ball is tossed straight up into the air from height 4 feet. What should be the initial velocity to have the ball reach a height of 70 feet?

13. Find the equation of the circle in the xy-plane centered at $(3, -2)$ with radius 7.

14. Find the equation of the circle in the xy-plane centered at $(-4, 5)$ with radius 6.

15. Find two choices for b such that $(5, b)$ is on the circle with radius 4 centered at $(3, 6)$.

16. Find two choices for b such that $(b, 4)$ is on the circle with radius 3 centered at $(-1, 6)$.

17. Find all numbers x such that
$$\frac{x-1}{x+3} = \frac{2x-1}{x+2}.$$

18. Find all numbers x such that
$$\frac{3x+2}{x-2} = \frac{2x-1}{x-1}.$$

19. Find two numbers w such that the points $(3, 1)$, $(w, 4)$, and $(5, w)$ all lie on a straight line.

20. Find two numbers r such that the points $(-1, 4)$, $(r, 2r)$, and $(1, r)$ all lie on a straight line.

21. The graph of the equation
$$x^2 - 6x + y^2 + 8y = -25$$
contains exactly one point. Find the coordinates of that point.

22. The graph of the equation
$$x^2 + 5x + y^2 - 3y = -\tfrac{17}{2}$$
contains exactly one point. Find the coordinates of that point.

23. Find the point on the line $y = 3x + 1$ in the xy-plane that is closest to the point $(2, 4)$.

24. Find the point on the line $y = 2x - 3$ in the xy-plane that is closest to the point $(5, 1)$.

25. Find a number t such that the distance between $(2, 3)$ and $(t, 2t)$ is as small as possible.

26. Find a number t such that the distance between $(-2, 1)$ and $(3t, 2t)$ is as small as possible.

27. Find the length of the graph of the curve defined by
$$y = \sqrt{9 - x^2}$$
with $-3 \le x \le 3$.

28. Find the length of the graph of the curve defined by
$$y = \sqrt{25 - x^2}$$
with $0 \le x \le 5$.

29. Find the two points where the circle of radius 2 centered at the origin intersects the circle of radius 3 centered at $(3, 0)$.

30. Find the two points where the circle of radius 3 centered at the origin intersects the circle of radius 4 centered at $(5, 0)$.

31. Find the equation of the circle in the xy-plane centered at the origin with circumference 9.

32. Find the equation of the circle in the xy-plane centered at $(3, 7)$ with circumference 5.

33. Find the equation of the circle centered at the origin in the uv-plane that has twice the circumference of the circle whose equation equals
$$u^2 + v^2 = 10.$$

34. Find the equation of the circle centered at the origin in the tw-plane that has three times the circumference of the circle whose equation equals
$$t^2 + w^2 = 5.$$

For Exercises 35 and 36, find the following information about the circles in the xy-plane described by the given equation:

(a) ***center***

(b) ***radius***

(c) ***diameter***

(d) ***circumference***

35. $x^2 - 8x + y^2 + 2y = -14$

36. $x^2 + 5x + y^2 - 6y = 3$

37. Find the intersection of the line containing the points $(2, 3)$ and $(4, 7)$ and the circle with radius $\sqrt{15}$ centered at $(3, -3)$.

38. Find the intersection of the line containing the points $(3, 4)$ and $(1, 8)$ and the circle with radius $\sqrt{3}$ centered at $(2, 9)$.

For Exercises 39–44, find the vertex of the graph of the given equation.

39. $y = 7x^2 - 12$

40. $y = -9x^2 - 5$

41. $y = (x - 2)^2 - 3$

42. $y = (x + 3)^2 + 4$

43. $y = (2x - 5)^2 + 6$

44. $y = (7x + 3)^2 + 5$

In Exercises 45–48, for the given equation:

(a) ***Write the right side of the equation in the form $k(x + t)^2 + r$.***

(b) ***Find the value of x where the right side of the equation attains its minimum value or its maximum value.***

(c) ***Find the minimum or maximum value of the right side of the equation.***

(d) ***Find the vertex of the graph of the equation.***

45. $y = x^2 + 7x + 12$

46. $y = 5x^2 + 2x + 1$

47. $y = -2x^2 + 5x - 2$

48. $y = -3x^2 + 5x - 1$

49. Find a constant c such that the graph of $y = x^2 + 6x + c$ has its vertex on the x-axis.

50. Find a constant c such that the graph of $y = x^2 + 5x + c$ in the xy-plane has its vertex on the line $y = x$.

51. Find two numbers whose sum equals 10 and whose product equals 7.

52. Find two numbers whose sum equals 6 and whose product equals 4.

53. Find two positive numbers whose difference equals 3 and whose product equals 20.

54. Find two positive numbers whose difference equals 4 and whose product equals 15.

55. Find the minimum value of $x^2 - 6x + 2$.

56. Find the minimum value of $3x^2 + 5x + 1$.

57. Find the maximum value of $7 - 2x - x^2$.

58. Find the maximum value of $9 + 5x - 4x^2$.

PROBLEMS

59. Show that
$$(a + b)^2 = a^2 + b^2$$
if and only if $a = 0$ or $b = 0$.

60. Explain why the graph of the equation
$$x^2 + 4x + y^2 - 10y = -30$$
contains no points.

61. Suppose
$$2x^2 + 3x + c > 0$$
for every real number x. Show that $c > \frac{9}{8}$.

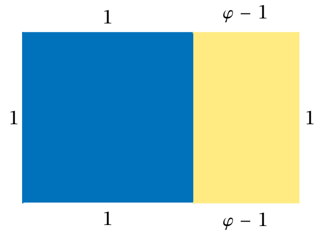
62. Suppose
$$3x^2 + bx + 7 > 0$$
for every real number x. Show that $|b| < 2\sqrt{21}$.

63. Suppose
$$at^2 + 5t + 4 > 0$$
for every real number t. Show that $a > \frac{25}{16}$.

64. Suppose $a \neq 0$ and $b^2 \geq 4ac$. Verify by direct substitution that if
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
then $ax^2 + bx + c = 0$.

65. Suppose $a \neq 0$ and $b^2 \geq 4ac$. Verify by direct calculation that
$$ax^2 + bx + c =$$
$$a\Big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\Big)\Big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\Big).$$

66. Find a number φ such that in the figure below, the yellow rectangle is similar to the large rectangle formed by the union of the blue square and the yellow rectangle.



The number φ that solves this problem is called the **golden ratio** (the symbol φ is the Greek letter phi). Rectangles whose ratio between the length of the long side and the length of the short side equals φ are supposedly the most aesthetically pleasing rectangles. The large rectangle formed by the union of the blue square and the yellow rectangle has the golden ratio, as does the yellow rectangle. Many works of art feature rectangles with the golden ratio.

67. Suppose a, b, and c are numbers with $a \neq 0$. Show that the vertex of the graph of
$$y = ax^2 + bx + c$$
is the point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

68. Suppose a, b, and c are numbers with $a \neq 0$ and that only one number t satisfies the equation
$$at^2 + bt + c = 0.$$
Show that t is the first coordinate of the vertex of the graph of $y = ax^2 + bx + c$ and that the second coordinate of the vertex equals 0.

69. Suppose a, b, and c are numbers such that exactly two real numbers t satisfy the equation $at^2 + bt + c = 0$. Show that the average of these two real numbers is the first coordinate of the vertex of the graph of $y = ax^2 + bx + c$.

70. Suppose b and c are numbers such that the equation
$$x^2 + bx + c = 0$$
has no real solutions. Explain why the equation
$$x^2 + bx - c = 0$$
has two real solutions.

71. Show that there do not exist two real numbers whose sum is 7 and whose product is 13.

72. Suppose $a > b > 0$. Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(\sqrt{a^2 - b^2}, 0)$.

73. Suppose $a > b > 0$. Find a formula in terms of x for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(-\sqrt{a^2 - b^2}, 0)$.

74. Suppose $a > b > 0$. Use the results of the two previous problems to show that $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ are foci of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

75. Suppose $b > a > 0$. Find a formula in terms of y for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, \sqrt{b^2 - a^2})$.

76. Suppose $b > a > 0$. Find a formula in terms of y for the distance from a typical point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the point $(0, -\sqrt{b^2 - a^2})$.

77. Suppose $b > a > 0$. Use the results of the two previous problems to show that $(0, \sqrt{b^2 - a^2})$ and $(0, -\sqrt{b^2 - a^2})$ are foci of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

78. Suppose a and b are nonzero numbers. Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ to the point $(0, -\sqrt{a^2 + b^2})$.

79. Suppose a and b are nonzero numbers. Find a formula in terms of y for the distance from a typical point (x, y) with $y > 0$ on the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ to the point $(0, \sqrt{a^2 + b^2})$.

80. Suppose a and b are nonzero numbers. Use the results of the two previous problems to show that $(0, -\sqrt{a^2 + b^2})$ and $(0, \sqrt{a^2 + b^2})$ are foci of the hyperbola
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

81. Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to the point $(-\sqrt{2}, -\sqrt{2})$ is $x + \frac{1}{x} + \sqrt{2}$. [*See Example 7 in Section 4.1 for a graph of* $y = \frac{1}{x}$.]

82. Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to the point $(\sqrt{2}, \sqrt{2})$ is $x + \frac{1}{x} - \sqrt{2}$.
83. Suppose $x > 0$. Show that the distance from $(x, \frac{1}{x})$ to $(-\sqrt{2}, -\sqrt{2})$ minus the distance from $(x, \frac{1}{x})$ to $(\sqrt{2}, \sqrt{2})$ equals $2\sqrt{2}$.
84. Explain why the result of the previous problem justifies calling the curve $y = \frac{1}{x}$ a hyperbola with foci at $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$.
85. Explain why `graph x^2/25 + y^2/9 = 1` in Wolfram|Alpha produces what appears to be a circle rather than a typical ellipse-shape as shown in Examples 8 and 9.
[*Hint:* Notice the scales on the two axes.]

WORKED-OUT SOLUTIONS *to Odd-numbered Exercises*

For Exercises 1–12, use the following information: If an object is thrown straight up into the air from height H feet at time 0 with initial velocity V feet per second, then at time t seconds the height of the object is

$$-16.1t^2 + Vt + H$$

feet. This formula uses only gravitational force, ignoring air friction. It is valid only until the object hits the ground or some other object.

Some notebook computers have a sensor that detects sudden changes in motion and stops the notebook's hard drive, protecting it from damage.

1. Suppose a notebook computer is accidentally knocked off a shelf that is six feet high. How long before the computer hits the ground?

SOLUTION In this case, we have $V = 0$ and $H = 6$ in the formula above. We want to know the time t at which the height equals 0. In other words, we need to solve the equation

$$-16.1t^2 + 6 = 0.$$

The solutions to the equation are $t = \pm\sqrt{\frac{6}{16.1}}$. The negative value makes no sense here, and thus $t = \sqrt{\frac{6}{16.1}} \approx 0.61$ seconds.

3. Suppose the motion detection/protection mechanism of a notebook computer takes 0.3 seconds to work after the computer starts to fall. What is the minimum height from which the notebook computer can fall and have the protection mechanism work?

SOLUTION We want to find the initial height H such that the notebook computer will hit the ground (meaning have height 0) after 0.3 seconds. In other words, we want

$$-16.1 \times 0.3^2 + H = 0.$$

Thus $H = 16.1 \times 0.3^2 = 1.449$ feet.

5. Suppose a ball is tossed straight up into the air from height 5 feet with initial velocity 20 feet per second.

(a) How long before the ball hits the ground?

(b) How long before the ball reaches its maximum height?

(c) What is the ball's maximum height?

SOLUTION

(a) The ball hits the ground when

$$-16.1t^2 + 20t + 5 = 0.$$

The quadratic formula shows that $t \approx 1.46$ seconds (the other solution produced by the quadratic formula has been discarded because it is negative).

(b) Completing the square, we have

$$\begin{aligned} -16.1t^2 &+ 20t + 5 \\ &= -16.1\Big[t^2 - \frac{20}{16.1}t\Big] + 5 \\ &= -16.1\Big[\Big(t - \frac{10}{16.1}\Big)^2 - \Big(\frac{10}{16.1}\Big)^2\Big] + 5 \\ &= -16.1\Big(t - \frac{10}{16.1}\Big)^2 + \frac{100}{16.1} + 5. \end{aligned}$$

The expression above gives the height of the ball at time t. Thus the ball reaches its maximum height when $t = \frac{10}{16.1} \approx 0.62$ seconds.

(c) The solution to part (b) shows that the maximum height of the ball is $\frac{100}{16.1} + 5 \approx 11.2$ feet.

7. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball stay in the air for 4 seconds?

SOLUTION Suppose the initial velocity of the ball is V. Then the height of the ball at time t is

$$-16.1t^2 + Vt + 5.$$

We want $t = 4$ when the ball hits the ground, meaning its height is 0. In other words, we want

$$-16.1 \times 4^2 + 4V + 5 = 0.$$

Solving this equation for V gives $V = 63.15$ feet per second.

9. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach its maximum height after 1 second?

SOLUTION Suppose the initial velocity of the ball is V. Then the height of the ball at time t is

$$\begin{aligned}
-16.1t^2 + Vt + 5 \\
&= -16.1\Big[t^2 - \frac{Vt}{16.1}\Big] + 5 \\
&= -16.1\Big[\Big(t - \frac{V}{32.2}\Big)^2 - \Big(\frac{V}{32.2}\Big)^2\Big] + 5 \\
&= -16.1\Big(t - \frac{V}{32.2}\Big)^2 + \frac{V^2}{64.4} + 5.
\end{aligned}$$

The ball reaches its maximum height when $t - \dfrac{V}{32.2} = 0$. We want this to happen when $t = 1$. In other words, we want $1 - \dfrac{V}{32.2} = 0$, which implies that $V = 32.2$ feet per second.

11. Suppose a ball is tossed straight up into the air from height 5 feet. What should be the initial velocity to have the ball reach a height of 50 feet?

SOLUTION Suppose the initial velocity of the ball is V. As can be seen from the solution to Exercise 9, the maximum height of the ball is $\dfrac{V^2}{64.4} + 5$, which we want to equal 50. Solving the equation

$$\frac{V^2}{64.4} + 5 = 50$$

for V gives $V \approx 53.83$ feet per second.

13. Find the equation of the circle in the xy-plane centered at $(3, -2)$ with radius 7.

SOLUTION The equation of this circle is

$$(x - 3)^2 + (y + 2)^2 = 49.$$

15. Find two choices for b such that $(5, b)$ is on the circle with radius 4 centered at $(3, 6)$.

SOLUTION The equation of the circle with radius 4 centered at $(3, 6)$ is

$$(x - 3)^2 + (y - 6)^2 = 16.$$

The point $(5, b)$ is on this circle if and only if

$$(5 - 3)^2 + (b - 6)^2 = 16,$$

which is equivalent to the equation $(b-6)^2 = 12$. Thus

$$b - 6 = \pm\sqrt{12} = \pm\sqrt{4 \cdot 3} = \pm\sqrt{4}\sqrt{3} = \pm 2\sqrt{3}.$$

Thus $b = 6 + 2\sqrt{3}$ or $b = 6 - 2\sqrt{3}$.

17. Find all numbers x such that

$$\frac{x - 1}{x + 3} = \frac{2x - 1}{x + 2}.$$

SOLUTION The equation above is equivalent to the equation

$$(x - 1)(x + 2) = (2x - 1)(x + 3).$$

Expanding and then collecting all terms on one side leads to the equation

$$x^2 + 4x - 1 = 0.$$

The quadratic formula now shows that $x = -2 \pm \sqrt{5}$.

19. Find two numbers w such that the points $(3, 1)$, $(w, 4)$, and $(5, w)$ all lie on a straight line.

SOLUTION The slope of the line containing $(3, 1)$ and $(w, 4)$ is $\frac{3}{w-3}$. The slope of the line containing $(3, 1)$ and $(5, w)$ is $\frac{w-1}{2}$. We need these two slopes to be equal, which means that

$$\frac{3}{w - 3} = \frac{w - 1}{2}.$$

Thus $2 \cdot 3 = (w - 1)(w - 3)$, which means that

$$w^2 - 4w - 3 = 0.$$

The quadratic formula now shows that $w = 2 \pm \sqrt{7}$.

21. The graph of the equation

$$x^2 - 6x + y^2 + 8y = -25$$

contains exactly one point. Find the coordinates of that point.

SOLUTION We complete the square to rewrite the equation above:

$$\begin{aligned} -25 &= x^2 - 6x + y^2 + 8y \\ &= (x-3)^2 - 9 + (y+4)^2 - 16 \\ &= (x-3)^2 + (y+4)^2 - 25. \end{aligned}$$

Thus the original equation can be rewritten as $(x-3)^2 + (y+4)^2 = 0$. The only solution to this equation is $x = 3$ and $y = -4$. Thus $(3, -4)$ is the only point on the graph of this equation.

23. Find the point on the line $y = 3x + 1$ in the xy-plane that is closest to the point $(2, 4)$.

SOLUTION A typical point on the line $y = 3x + 1$ in the xy-plane has coordinates $(x, 3x+1)$. The distance between this point and $(2, 4)$ equals

$$\sqrt{(x-2)^2 + (3x+1-4)^2},$$

which with a bit of algebra can be rewritten as

$$\sqrt{10x^2 - 22x + 13}.$$

We want to make the quantity above as small as possible, which means that we need to make $10x^2 - 22x$ as small as possible. This can be done by completing the square:

$$\begin{aligned} 10x^2 - 22x &= 10\Big[x^2 - \frac{11}{5}x\Big] \\ &= 10\Big[\Big(x - \frac{11}{10}\Big)^2 - \frac{121}{100}\Big]. \end{aligned}$$

The last quantity will be as small as possible when $x = \frac{11}{10}$. Plugging $x = \frac{11}{10}$ into the equation $y = 3x + 1$ gives $y = \frac{43}{10}$. Thus $(\frac{11}{10}, \frac{43}{10})$ is the point on the line $y = 3x + 1$ that is closest to the point $(2, 4)$.

25. Find a number t such that the distance between $(2, 3)$ and $(t, 2t)$ is as small as possible.

SOLUTION The distance between $(2, 3)$ and $(t, 2t)$ equals

$$\sqrt{(t-2)^2 + (2t-3)^2}.$$

We want to make this as small as possible, which happens when

$$(t-2)^2 + (2t-3)^2$$

is as small as possible. Note that

$$(t-2)^2 + (2t-3)^2 = 5t^2 - 16t + 13.$$

This will be as small as possible when $5t^2 - 16t$ is as small as possible. To find when that happens, we complete the square:

$$\begin{aligned} 5t^2 - 16t &= 5\Big[t^2 - \tfrac{16}{5}t\Big] \\ &= 5\Big[\Big(t - \tfrac{8}{5}\Big)^2 - \tfrac{64}{25}\Big]. \end{aligned}$$

This quantity is made smallest when $t = \frac{8}{5}$.

27. Find the length of the graph of the curve defined by

$$y = \sqrt{9 - x^2}$$

with $-3 \le x \le 3$.

SOLUTION Squaring both sides of the equation $y = \sqrt{9 - x^2}$ and then adding x^2 to both sides gives the equation $x^2 + y^2 = 9$, which is the equation of the circle of radius 3 centered at the origin. However, the equation $y = \sqrt{9 - x^2}$ implies that $y \ge 0$, and thus we have only the top half of the circle.

The entire circle of radius 3 has circumference 6π. Thus the graph of $y = \sqrt{9 - x^2}$, which is half of the circle, has length 3π.

29. Find the two points where the circle of radius 2 centered at the origin intersects the circle of radius 3 centered at $(3, 0)$.

SOLUTION The equations of these two circles are

$$x^2 + y^2 = 4 \quad \text{and} \quad (x-3)^2 + y^2 = 9.$$

Subtracting the first equation from the second equation, we get

$$(x-3)^2 - x^2 = 5,$$

which simplifies to the equation $-6x + 9 = 5$, whose solution is $x = \frac{2}{3}$. Plugging this value of x into either of the equations above and solving for y gives $y = \pm\frac{4\sqrt{2}}{3}$. Thus the two circles intersect at the points $(\frac{2}{3}, \frac{4\sqrt{2}}{3})$ and $(\frac{2}{3}, -\frac{4\sqrt{2}}{3})$.

31. Find the equation of the circle in the xy-plane centered at the origin with circumference 9.

SOLUTION Let r denote the radius of this circle. Then $2\pi r = 9$, which implies that $r = \frac{9}{2\pi}$. Thus the equation of the circle is

$$x^2 + y^2 = \frac{81}{4\pi^2}.$$

33. Find the equation of the circle centered at the origin in the uv-plane that has twice the circumference of the circle whose equation equals

$$u^2 + v^2 = 10.$$

SOLUTION The equation given above describes a circle centered at the origin whose radius equals $\sqrt{10}$. Because the circumference is proportional to the radius, if we want a circle with twice the circumference then we need to double the radius. Thus the circle we seek has radius $2\sqrt{10}$. Because $(2\sqrt{10})^2 = 2^2 \cdot \sqrt{10}^2 = 40$, the equation we seek is

$$u^2 + v^2 = 40.$$

For Exercises 35 and 36, find the following information about the circles in the xy-plane described by the given equation:

(a) ***center*** **(c)** ***diameter***

(b) ***radius*** **(d)** ***circumference***

35. $x^2 - 8x + y^2 + 2y = -14$

SOLUTION Completing the square, we can rewrite the left side of this equation as follows:

$$\begin{aligned} x^2 - 8x + y^2 + 2y &= (x-4)^2 - 16 + (y+1)^2 - 1 \\ &= (x-4)^2 + (y+1)^2 - 17. \end{aligned}$$

Substituting this expression into the left side of the original equation and then adding 17 to both sides shows that the original equation is equivalent to the equation

$$(x-4)^2 + (y+1)^2 = 3.$$

(a) The equation above shows that this circle has center $(4, -1)$.

(b) The equation above shows that this circle has radius $\sqrt{3}$.

(c) Because the diameter is twice the radius, this circle has diameter $2\sqrt{3}$.

(d) Because the circumference is 2π times the radius, this circle has circumference $2\pi\sqrt{3}$.

37. Find the intersection of the line containing the points $(2, 3)$ and $(4, 7)$ and the circle with radius $\sqrt{15}$ centered at $(3, -3)$.

SOLUTION First we find the equation of the line containing the points $(2, 3)$ and $(4, 7)$. This line will have slope $\frac{7-3}{4-2}$, which equals 2. Thus the equation of this line will have the form $y = 2x + b$. Because $(2, 3)$ is on this line, we can substitute $x = 2$ and $y = 3$ into the last equation and then solve for b, getting $b = -1$. Thus the equation of the line containing the points $(2, 3)$ and $(4, 7)$ is

$$y = 2x - 1.$$

The equation of the circle with radius $\sqrt{15}$ centered at $(3, -3)$ is

$$(x-3)^2 + (y+3)^2 = 15.$$

To find the intersection of the circle and the line, we replace y by $2x - 1$ in the equation above, getting

$$(x-3)^2 + (2x+2)^2 = 15.$$

Expanding the terms in the equation above and then collecting terms gives the equation

$$5x^2 + 2x - 2 = 0.$$

Using the quadratic formula, we then find that

$$x = \frac{-1+\sqrt{11}}{5} \quad \text{or} \quad x = \frac{-1-\sqrt{11}}{5}.$$

Substituting these values of x into the equation $y = 2x - 1$ shows that the line intersects the circle in the points

$$\Big(\frac{-1+\sqrt{11}}{5}, \frac{-7+2\sqrt{11}}{5}\Big)$$

and

$$\Big(\frac{-1-\sqrt{11}}{5}, \frac{-7-2\sqrt{11}}{5}\Big).$$

For Exercises 39–44, find the vertex of the graph of the given equation.

39. $y = 7x^2 - 12$

SOLUTION The value of $7x^2 - 12$ is minimized when $x = 0$. When $x = 0$, the value of $7x^2 - 12$ equals -12. Thus the vertex of the graph of $y = 7x^2 - 12$ is $(0, -12)$.

41. $y = (x-2)^2 - 3$

SOLUTION The value of $(x-2)^2 - 3$ is minimized when $x - 2 = 0$. When $x = 2$, the value of $(x-2)^2 - 3$ equals -3. Thus the vertex of the graph of $y = (x-2)^2 - 3$ is $(2, -3)$.

43. $y = (2x-5)^2 + 6$

SOLUTION The value of $(2x-5)^2 + 6$ is minimized when $2x - 5 = 0$. This happens when $x = \frac{5}{2}$, which makes the value of $(2x-5)^2 + 6$ equal to 6. Thus the vertex of the graph of $y = (2x-5)^2 + 6$ is $(\frac{5}{2}, 6)$.

In Exercises 45–48, for the given equation:

(a) ***Write the right side of the equation in the form $k(x+t)^2 + r$.***

(b) ***Find the value of x where the right side of the equation attains its minimum value or its maximum value.***

(c) ***Find the minimum or maximum value of the right side of the equation.***

(d) ***Find the vertex of the graph of the equation.***

45. $y = x^2 + 7x + 12$

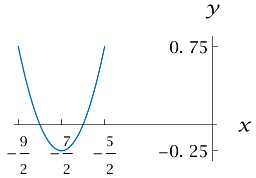
SOLUTION

(a) By completing the square, we can write

$$\begin{aligned} x^2 + 7x + 12 &= (x + \tfrac{7}{2})^2 - \tfrac{49}{4} + 12 \\ &= (x + \tfrac{7}{2})^2 - \tfrac{1}{4}. \end{aligned}$$

(b) The expression above shows that the value of the right side is minimized when $x = -\frac{7}{2}$.

(c) The solution to part (a) shows that the minimum value of this expression is $-\frac{1}{4}$, which occurs when $x = -\frac{7}{2}$.



(d) The figure above shows that the vertex of the graph of $y = x^2 + 7x + 12$ is the point $(-\frac{7}{2}, -\frac{1}{4})$, as follows from the solutions to parts (b) and (c) and as is shown in the graph above.

47. $y = -2x^2 + 5x - 2$

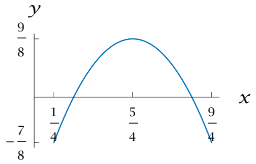
SOLUTION

(a) By completing the square, we can write

$$\begin{aligned} -2x^2 + 5x - 2 &= -2[x^2 - \tfrac{5}{2}x] - 2 \\ &= -2[(x - \tfrac{5}{4})^2 - \tfrac{25}{16}] - 2 \\ &= -2(x - \tfrac{5}{4})^2 + \tfrac{25}{8} - 2 \\ &= -2(x - \tfrac{5}{4})^2 + \tfrac{9}{8}. \end{aligned}$$

(b) The expression above shows that the value of $-2x^2 + 5x - 2$ is maximized when $x = \frac{5}{4}$.

(c) The solution to part (a) shows that the maximum value of this expression is $\frac{9}{8}$, which occurs when $x = \frac{5}{4}$.



(d) The figure above shows that the vertex of the graph of $y = -2x^2 + 5x - 2$ is the point $(\frac{5}{4}, \frac{9}{8})$, as follows from the solutions to parts (b) and (c) and as is shown in the graph above.

49. Find a constant c such that the graph of $y = x^2 + 6x + c$ has its vertex on the x-axis.

SOLUTION First we find the vertex of the graph of $y = x^2 + 6x + c$. To do this, complete the square:

$$x^2 + 6x + c = (x+3)^2 - 9 + c.$$

Thus the value of $x^2 + 6x + c$ is minimized when $x = -3$. When $x = -3$, the value of $x^2 + 6x + c$

equals $-9+c$. Thus the vertex of $y = x^2+6x+c$ is $(-3, -9+c)$.

The x-axis consists of the points whose second coordinate equals 0. Thus the vertex of the graph of $y = x^2+6x+c$ will be on the x-axis when $-9+c = 0$, or equivalently when $c = 9$.

51. Find two numbers whose sum equals 10 and whose product equals 7.

SOLUTION Let's call the two numbers s and t. We want

$$s+t = 10 \quad \text{and} \quad st = 7.$$

Solving the first equation for s, we have $s = 10-t$. Substituting this expression for s into the second equation gives $(10-t)t = 7$, which is equivalent to the equation

$$t^2 - 10t + 7 = 0.$$

Using the quadratic formula to solve this equation for t gives

$$\begin{aligned} t = \frac{10 \pm \sqrt{10^2 - 4\cdot 7}}{2} &= \frac{10 \pm \sqrt{72}}{2} \\ &= \frac{10 \pm \sqrt{36\cdot 2}}{2} = 5 \pm 3\sqrt{2}. \end{aligned}$$

Let's choose the solution $t = 5+3\sqrt{2}$. Plugging this value of t into the equation $s = 10-t$ then gives $s = 5-3\sqrt{2}$.

Thus two numbers whose sum equals 10 and whose product equals 7 are $5-3\sqrt{2}$ and $5+3\sqrt{2}$.

CHECK To check that this solution is correct, note that

$$(5-3\sqrt{2}) + (5+3\sqrt{2}) = 10$$

and

$$(5-3\sqrt{2})(5+3\sqrt{2}) = 5^2 - 3^2\sqrt{2}^2 = 25 - 9\cdot 2 = 7.$$

53. Find two positive numbers whose difference equals 3 and whose product equals 20.

SOLUTION Let's call the two numbers s and t. We want

$$s-t = 3 \quad \text{and} \quad st = 20.$$

Solving the first equation for s, we have $s = t+3$. Substituting this expression for s into the second equation gives $(t+3)t = 20$, which is equivalent to the equation

$$t^2 + 3t - 20 = 0.$$

Using the quadratic formula to solve this equation for t gives

$$t = \frac{-3 \pm \sqrt{3^2 + 4\cdot 20}}{2} = \frac{-3 \pm \sqrt{89}}{2}.$$

Choosing the minus sign in the plus-or-minus expression above would lead to a negative value for t. Because this exercise requires that t be positive, we choose $t = \frac{-3+\sqrt{89}}{2}$. Plugging this value of t into the equation $s = t+3$ then gives $s = \frac{3+\sqrt{89}}{2}$.

Thus two numbers whose difference equals 3 and whose product equals 20 are $\frac{3+\sqrt{89}}{2}$ and $\frac{-3+\sqrt{89}}{2}$.

CHECK To check that this solution is correct, note that

$$\frac{3+\sqrt{89}}{2} - \frac{-3+\sqrt{89}}{2} = \frac{6}{2} = 3$$

and

$$\begin{aligned} \frac{3+\sqrt{89}}{2} \cdot \frac{-3+\sqrt{89}}{2} &= \frac{\sqrt{89}+3}{2} \cdot \frac{\sqrt{89}-3}{2} \\ &= \frac{\sqrt{89}^2 - 3^2}{4} = \frac{80}{4} = 20. \end{aligned}$$

55. Find the minimum value of $x^2 - 6x + 2$.

SOLUTION By completing the square, we can write

$$\begin{aligned} x^2 - 6x + 2 &= (x-3)^2 - 9 + 2 \\ &= (x-3)^2 - 7. \end{aligned}$$

The expression above shows that the minimum value of $x^2 - 6x + 2$ is -7 (and that this minimum value occurs when $x = 3$).

57. Find the maximum value of $7 - 2x - x^2$.

SOLUTION By completing the square, we can write

$$\begin{aligned} 7 - 2x - x^2 &= -[x^2 + 2x] + 7 \\ &= -[(x+1)^2 - 1] + 7 \\ &= -(x+1)^2 + 8. \end{aligned}$$

The expression above shows that the maximum value of $7 - 2x - x^2$ is 8 (and that this maximum value occurs when $x = -1$).

2.4 *Area*

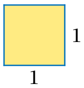
LEARNING OBJECTIVES

By the end of this section you should be able to

- compute the areas of squares, rectangles, parallelograms, triangles, and trapezoids;
- explain how area changes when the coordinate axes are stretched;
- compute the area inside a circle;
- compute the area inside an ellipse.

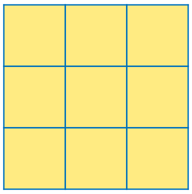
You probably already have a good intuitive notion of area. In this section we will try to strengthen this intuition and build a good understanding of the formulas for the area of simple regions.

Squares, Rectangles, and Parallelograms



A 1-by-1 square.

The most primitive notion of area is that a 1-by-1 square has area 1. If we can decompose a region into 1-by-1 squares, then the area of that region is the number of 1-by-1 squares into which it can be decomposed, as shown in the figure below:

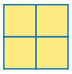


A 3-by-3 square can be decomposed into nine 1-by-1 squares. Thus a 3-by-3 square has area 9.

The expression m^2 is called "m squared" because a square whose sides have length m has area m^2.

If m is a positive integer, then an m-by-m square can be decomposed into m^2 squares of size 1-by-1. Thus it is no surprise that the area of an m-by-m square equals m^2.

The same formula holds for squares whose side length is not necessarily an integer, as shown below:

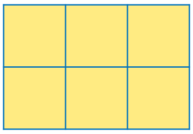


Four $\frac{1}{2}$-by-$\frac{1}{2}$ squares fill up a 1-by-1 square. Thus each $\frac{1}{2}$-by-$\frac{1}{2}$ square has area $\frac{1}{4}$.

More generally, we have the following formula:

Area of a square

A square whose sides have length ℓ has area ℓ^2.



Consider a rectangle with base 3 and height 2, as shown here. This 3-by-2 rectangle can be decomposed into six 1-by-1 squares. Thus this rectangle has area 6.

Similarly, if b and h are positive integers, then a rectangle with base b and height h can be composed into bh squares of size 1-by-1, showing that the rectangle has area bh. More generally, the same formula is valid even if the base and height are not integers.

Use the same units for the base and the height. The unit of measurement for area is then the square of the unit used for these lengths. For example, a rectangle with base 3 feet and height 2 feet has area 6 square feet.

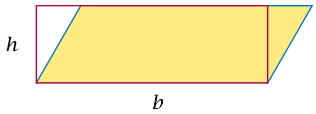
Area of a rectangle

A rectangle with base b and height h has area bh.

In the special case where the base equals the height, the formula for the area of a rectangle becomes the formula for the area of a square.

A **parallelogram** is a quadrilateral (a four-sided polygon) in which both pairs of opposite sides are parallel, as shown here.

To find the area of a parallelogram, select one of the sides and call its length the **base.** The opposite side of the parallelogram will have the same length. The **height** of the parallelogram is then defined to be the length of a line segment that connects these two sides and is perpendicular to both of them. Thus in the figure shown here, the parallelogram has base b and both vertical line segments have length equal to the height h.



The yellow region is a parallelogram with base b and height h. The area of the parallelogram is the same as the area of the rectangle (outlined in red) with base b and height h.

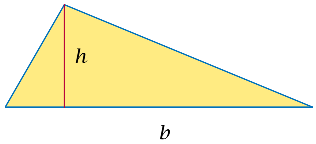
The two small triangles in the figure above have the same size and thus the same area. The rectangle in the figure above could be obtained from the parallelogram by moving the triangle on the right to the position of the triangle on the left. This shows that the parallelogram and the rectangle above have the same area. Because the area of the rectangle equals bh, we thus have the following formula for the area of a parallelogram:

Area of a parallelogram

A parallelogram with base b and height h has area bh.

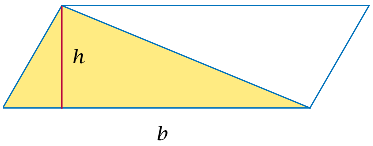
Triangles and Trapezoids

To find the area of a triangle, select one of the sides and call its length the **base.** The **height** of the triangle is then defined to be the length of the perpendicular line segment that connects the opposite vertex to the side determining the base, as shown in the figure in the margin.



A triangle with base b and height h.

To derive the formula for the area of a triangle with base b and height h, draw two line segments, each parallel to and the same length as one of the sides of the triangle, to form a parallelogram as in the figure below:



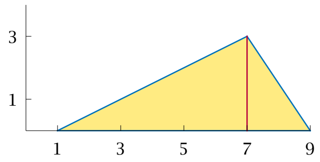
The triangle has been extended to a parallelogram by adjoining a second triangle with the same side lengths as the original triangle.

The parallelogram above has base b and height h and hence has area bh. The original triangle has area equal to half the area of the parallelogram. Thus we obtain the following formula:

Area of a triangle

A triangle with base b and height h has area $\frac{1}{2}bh$.

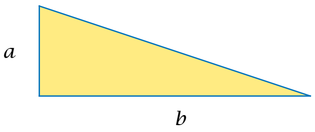
EXAMPLE 1 Find the area of the triangle whose vertices are $(1, 0)$, $(9, 0)$, and $(7, 3)$.



SOLUTION Choose the side connecting $(1, 0)$ and $(9, 0)$ as the base of this triangle. Thus this triangle has base $9 - 1$, which equals 8.

The height of this triangle is the length of the red line shown here; this height equals the second coordinate of the vertex $(7, 3)$. In other words, this triangle has height 3.

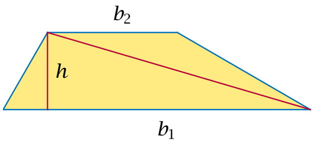
Thus this triangle has area $\frac{1}{2} \cdot 8 \cdot 3$, which equals 12.



A right triangle with area $\frac{1}{2}ab$.

Consider the special case where our triangle happens to be a right triangle, with the right angle between sides of length a and b. Choosing b to be the base of the triangle, we see that the height of this triangle equals a. Thus in this case the area of the triangle equals $\frac{1}{2}ab$.

A **trapezoid** is a quadrilateral that has at least one pair of parallel sides, as for example shown here. The lengths of a pair of opposite parallel sides are called the **bases**, which are denoted below by b_1 and b_2. The **height** of the trapezoid, denoted h below, is then defined to be the length of a line segment that connects these two sides and that is perpendicular to both of them.



A trapezoid with bases b_1 and b_2 and height h.

The diagonal in the figure here divides the trapezoid into two triangles. The lower triangle has base b_1 and height h; thus the lower triangle has area $\frac{1}{2}b_1h$. The upper triangle has base b_2 and height h; thus the upper triangle has area $\frac{1}{2}b_2h$. The area of the trapezoid is the sum of the areas of these two triangles. Thus the area of the trapezoid equals $\frac{1}{2}b_1h + \frac{1}{2}b_2h$. Factoring out the $\frac{1}{2}$ and the h in this expression gives the following formula:

Area of a trapezoid

A trapezoid with bases b_1, b_2 and height h has area $\frac{1}{2}(b_1 + b_2)h$.

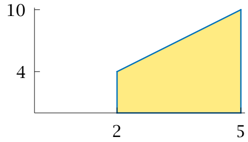
Note that $\frac{1}{2}(b_1 + b_2)$ is just the average of the two bases of the trapezoid. In the special case where the trapezoid is a parallelogram, the two bases are equal and we are back to the familiar formula that the area of a parallelogram equals the base times the height.

EXAMPLE 2

Find the area of the region in the xy-plane under the line $y = 2x$, above the x-axis, and between the lines $x = 2$ and $x = 5$.

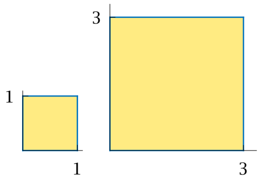
SOLUTION

The line $x = 2$ intersects the line $y = 2x$ at the point $(2, 4)$. The line $x = 5$ intersects the line $y = 2x$ at the point $(5, 10)$.

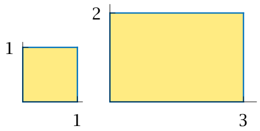


Thus the region in question is the trapezoid shown above. The parallel sides of this trapezoid (the two vertical sides) have lengths 4 and 10, and thus this trapezoid has bases 4 and 10. As can be seen from the figure above, this trapezoid has height 3 (note that in this trapezoid, the height is the length of the horizontal side). Thus the area of this trapezoid is $\frac{1}{2} \cdot (4 + 10) \cdot 3$, which equals 21.

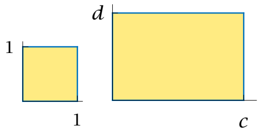
Stretching



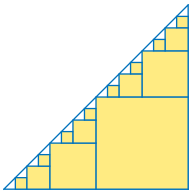
Suppose a square whose sides have length 1 has its sides tripled in length, resulting in a square whose sides have length 3, as shown here. You can think of this transformation as stretching both vertically and horizontally by a factor of 3. This transformation increases the area of the square by a factor of 9.



Consider now the transformation that stretches horizontally by a factor of 3 and stretches vertically by a factor of 2. This transformation changes a square whose sides have length 1 into a rectangle with base 3 and height 2, as shown here. Thus the area has been increased by a factor of 6.



More generally, suppose c, d are positive numbers, and consider the transformation that stretches horizontally by a factor of c and stretches vertically by a factor of d. This transformation changes a square whose sides have length 1 into a rectangle with base c and height d, as shown here. Thus the area has been increased by a factor of cd.



We need not restrict our attention to squares. The transformation that stretches horizontally by a factor of c and stretches vertically by a factor of d will change any region into a new region whose area has been changed by a factor of cd. This result follows from the result for squares, because any region can be approximated by a union of squares, as shown here for a triangle. Here is the formal statement of this result:

Area Stretch Theorem

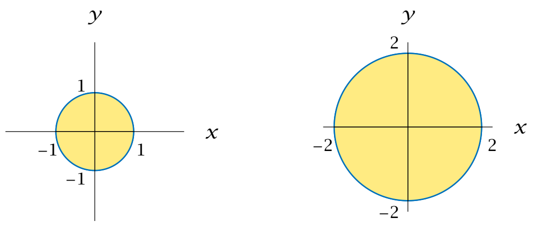
Suppose R is a region in the coordinate plane and c, d are positive numbers. Let R' be the region obtained from R by stretching horizontally by a factor of c and stretching vertically by a factor of d. Then

the area of R' equals cd times the area of R.

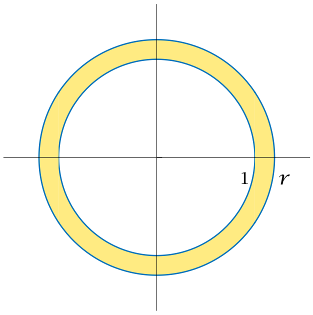
Circles and Ellipses

We will now derive the formula for the area inside a circle of radius r. You are surely already familiar with this formula. The goal here is to explain why it is true.

Consider the region inside a circle of radius 1 centered at the origin. If we stretch both horizontally and vertically by a factor of r, this region becomes the region inside the circle of radius r centered at the origin, as shown in the figure below for $r = 2$.



Stretching both horizontally and vertically by a factor of 2 *transforms a circle of radius* 1 *into a circle of radius* 2*.*



Let p denote the area inside a circle of radius 1. The Area Stretch Theorem implies that the area inside a circle of radius r equals $r^2 p$, which we write in the more familiar form pr^2. We need to find the value of p.

To find p, consider a circle of radius 1 surrounded by a slightly larger circle with radius r, as shown in the margin. Cut out the region between the two circles, then cut a slit in it and unwind it into the shape of a trapezoid (this requires a tiny bit of distortion) as shown below.

The upper base of the trapezoid is the circumference of the circle of radius 1*; the bottom base is the circumference of the circle of radius r.*

The trapezoid has height $r - 1$, which is the distance between the two original circles. The trapezoid has bases $2\pi r$ and 2π, corresponding to the circumferences of the two circles. Thus the trapezoid has area

$$\tfrac{1}{2}(2\pi r + 2\pi)(r - 1),$$

which equals $\pi(r + 1)(r - 1)$, which equals $\pi(r^2 - 1)$.

The area inside the larger circle equals the area inside the circle of radius 1 plus the area of the region between the two circles. In other words, the area inside the larger circle equals $p + \pi(r^2 - 1)$. The area inside the larger circle also equals pr^2, because the larger circle has radius r. Thus we have

Our derivation of the formula for the area inside a circle shows the intimate connection between the area and the circumference of a circle.

$$pr^2 = p + \pi(r^2 - 1).$$

Subtracting p from both sides, we get

$$p(r^2 - 1) = \pi(r^2 - 1).$$

Thus $p = \pi$. In other words, the area inside a circle of radius r equals πr^2.

We have derived the following formula:

Area inside a circle

The area inside a circle of radius r is πr^2.

Pie and π: The area of this pie with radius 4 inches is 16π square inches.

Thus to find the area inside a circle, we must first find the radius of the circle. Finding the radius sometimes requires a preliminary algebraic manipulation such as completing the square, as shown in the following example.

EXAMPLE 3

Consider the circle described by the equation

$$x^2 - 8x + y^2 + 6y = 4.$$

(a) Find the center of this circle.

(b) Find the radius of this circle.

(c) Find the circumference of this circle.

(d) Find the area inside this circle.

SOLUTION To obtain the desired information about the circle, we put its equation in a standard form. This can be done by completing the square:

$$\begin{aligned} 4 &= x^2 - 8x + y^2 + 6y \\ &= (x-4)^2 - 16 + (y+3)^2 - 9 \\ &= (x-4)^2 + (y+3)^2 - 25. \end{aligned}$$

Adding 25 to the first and last sides above shows that the circle is described by the equation

$$(x-4)^2 + (y+3)^2 = 29.$$

(a) The equation above shows that the center of the circle is $(4, -3)$.

(b) The equation above shows that the radius of the circle is $\sqrt{29}$.

(c) Because the circle has radius $\sqrt{29}$, its circumference is $2\sqrt{29}\pi$.

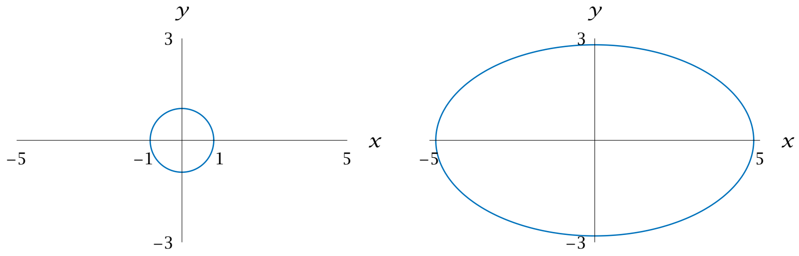
(d) Because the circle has radius $\sqrt{29}$, its area is 29π.

Do not make the mistake of thinking that this circle has radius 29.

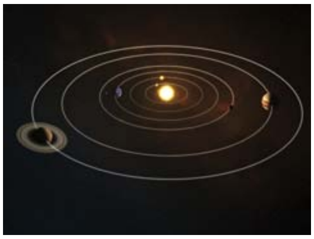
In Section 2.3 we saw that the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

is obtained from the circle of radius 1 centered at the origin by stretching horizontally by a factor of 5 and stretching vertically by factor of 3.



Stretching horizontally by a factor of 5 *and vertically by a factor of* 3 *transforms the circle on the left into the ellipse on the right.*



In addition to discovering that the orbits of planets are ellipses, Kepler also discovered that a line joining a planet to the sun sweeps out equal areas in equal times.

Because $5 \cdot 3 = 15$, the Area Stretch Theorem tells us that the area inside this ellipse equals 15 times the area inside the circle of radius 1. Because the area inside a circle of radius 1 is π, we conclude that the area inside this ellipse is 15π.

More generally, suppose a and b are positive numbers. Suppose the circle of radius 1 centered at the origin is stretched horizontally by a factor of a and stretched vertically by a factor of b. As we saw in Section 2.3, the equation of the resulting ellipse in the xy-plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The Area Stretch Theorem now gives us the following formula:

Area inside an ellipse

Suppose a and b are positive numbers. Then the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

EXAMPLE 4 Find the area inside the ellipse

$$4x^2 + 5y^2 = 3.$$

SOLUTION To put the equation of this ellipse in the form given by the area formula, begin by dividing both sides by 3, and then force the equation into the desired form, as follows:

$$\begin{aligned} 1 &= \tfrac{4}{3}x^2 + \tfrac{5}{3}y^2 \\ &= \frac{x^2}{\frac{3}{4}} + \frac{y^2}{\frac{3}{5}} \\ &= \frac{x^2}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{y^2}{\sqrt{\frac{3}{5}}^2}. \end{aligned}$$

Thus the area inside the ellipse is $\pi \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{\frac{3}{5}}$, which equals $\frac{3\sqrt{5}}{10}\pi$.

Ellipses need not be centered at the origin. For example, the equation

$$\frac{(x-5)^2}{9} + \frac{(y-7)^2}{16} = 1$$

represents an ellipse centered at a point $(5, 7)$. This ellipse is obtained by shifting the ellipse whose equation is $\frac{x^2}{9} + \frac{y^2}{16} = 1$ right 5 units and up 7 units. The formula above tells us that the area inside the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ is 12π, and thus the area inside the ellipse $\frac{(x-5)^2}{9} + \frac{(y-7)^2}{16} = 1$ is also 12π.

More generally, if a and b are positive numbers, then the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

represents an ellipse centered at a point (h, k). This ellipse is obtained by shifting the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Thus the area inside the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is πab.

"The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word."
—Galileo

EXERCISES

1. Find the area of a triangle that has two sides of length 6 and one side of length 10.

2. Find the area of a triangle that has two sides of length 6 and one side of length 4.

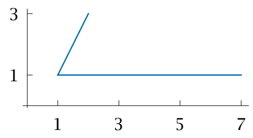
3. (a) Find the distance from the point $(2, 3)$ to the line containing the points $(-2, -1)$ and $(5, 4)$.
   (b) Use the information from part (a) to find the area of the triangle whose vertices are $(2, 3)$, $(-2, -1)$, and $(5, 4)$.

4. (a) Find the distance from the point $(3, 4)$ to the line containing the points $(1, 5)$ and $(-2, 2)$.
   (b) Use the information from part (a) to find the area of the triangle whose vertices are $(3, 4)$, $(1, 5)$, and $(-2, 2)$.

5. Find the area of the triangle whose vertices are $(2, 0)$, $(9, 0)$, and $(4, 5)$.

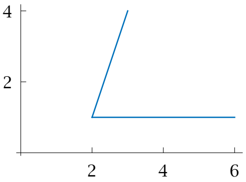
6. Find the area of the triangle whose vertices are $(-3, 0)$, $(2, 0)$, and $(4, 3)$.

7. Suppose $(2, 3)$, $(1, 1)$, and $(7, 1)$ are three vertices of a parallelogram, two of whose sides are shown here.



   (a) Find the fourth vertex of this parallelogram.
   (b) Find the area of this parallelogram.

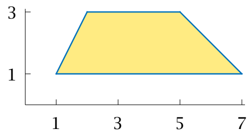
8. Suppose $(3, 4)$, $(2, 1)$, and $(6, 1)$ are three vertices of a parallelogram, two of whose sides are shown here.



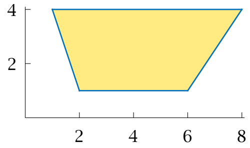
(a) Find the fourth vertex of this parallelogram.

(b) Find the area of this parallelogram.

9. Find the area of this trapezoid, whose vertices are $(1, 1)$, $(7, 1)$, $(5, 3)$, and $(2, 3)$.



10. Find the area of this trapezoid, whose vertices are $(2, 1)$, $(6, 1)$, $(8, 4)$, and $(1, 4)$.



11. Find the area of the region in the xy-plane under the line $y = \frac{x}{2}$, above the x-axis, and between the lines $x = 2$ and $x = 6$.

12. Find the area of the region in the xy-plane under the line $y = 3x + 1$, above the x-axis, and between the lines $x = 1$ and $x = 5$.

13. Let $f(x) = |x|$. Find the area of the region in the xy-plane under the graph of f, above the x-axis, and between the lines $x = -2$ and $x = 5$.

14. Let $f(x) = |2x|$. Find the area of the region in the xy-plane under the graph of f, above the x-axis, and between the lines $x = -3$ and $x = 4$.

15. Find the area inside a circle with diameter 7.

16. Find the area inside a circle with diameter 9.

17. Find the area inside a circle with circumference 5 feet.

18. Find the area inside a circle with circumference 7 yards.

19. Find the area inside the circle whose equation is
$$x^2 - 6x + y^2 + 10y = 1.$$

20. Find the area inside the circle whose equation is
$$x^2 + 5x + y^2 - 3y = 1.$$

21. Find a number t such that the area inside the circle
$$3x^2 + 3y^2 = t$$
is 8.

22. Find a number t such that the area inside the circle
$$5x^2 + 5y^2 = t$$
is 2.

Use the following information for Exercises 23–28: A standard DVD disk has a 12-cm diameter (cm is the abbreviation for centimeters). The hole in the center of the disk has a 0.75-cm radius. About 50.2 megabytes of data can be stored on each square cm of usable surface of the DVD disk.

23. What is the area of a DVD disk, not counting the hole?

24. What is the area of a DVD disk, not counting the hole and the unusable circular ring with width 1.5 cm that surrounds the hole?

25. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 3100 megabytes of data storage?

26. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 4200 megabytes of data storage?

27. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 200 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?

28. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 350 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?

29. Find the area of the region in the xy-plane under the curve $y = \sqrt{4 - x^2}$ (with $-2 \le x \le 2$) and above the x-axis.

30. Find the area of the region in the xy-plane under the curve $y = \sqrt{9 - x^2}$ (with $-3 \le x \le 3$) and above the x-axis.

31. Using the answer from Exercise 29, find the area of the region in the xy-plane under the curve $y = 3\sqrt{4 - x^2}$ (with $-2 \le x \le 2$) and above the x-axis.

32. Using the answer from Exercise 30, find the area of the region in the xy-plane under the curve $y = 5\sqrt{9 - x^2}$ (with $-3 \le x \le 3$) and above the x-axis.

33. Using the answer from Exercise 29, find the area of the region in the xy-plane under the curve $y = \sqrt{4 - \frac{x^2}{9}}$ (with $-6 \le x \le 6$) and above the x-axis.

34. Using the answer from Exercise 30, find the area of the region in the xy-plane under the curve $y = \sqrt{9 - \frac{x^2}{16}}$ (with $-12 \le x \le 12$) and above the x-axis.

35. Find the area of the region in the xy-plane under the curve

$$y = 1 + \sqrt{4 - x^2},$$

above the x-axis, and between the lines $x = -2$ and $x = 2$.

36. Find the area of the region in the xy-plane under the curve

$$y = 2 + \sqrt{9 - x^2},$$

above the x-axis, and between the lines $x = -3$ and $x = 3$.

In Exercises 37–44, find the area inside the ellipse in the xy-plane determined by the given equation.

37. $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$

38. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

39. $2x^2 + 3y^2 = 1$

40. $10x^2 + 7y^2 = 1$

41. $3x^2 + 2y^2 = 7$

42. $5x^2 + 9y^2 = 3$

43. $3x^2 + 4x + 2y^2 + 3y = 2$

44. $4x^2 + 2x + 5y^2 + y = 2$

45. Find a positive number c such that the area inside the ellipse

$$2x^2 + cy^2 = 5$$

is 3.

46. Find a positive number c such that the area inside the ellipse

$$cx^2 + 7y^2 = 3$$

is 2.

47. Find numbers a and b such that $a > b$, $a + b = 15$, and the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is 36π.

48. Find numbers a and b such that $a > b$, $a + b = 5$, and the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is 3π.

49. Find a number t such that the area inside the ellipse

$$4x^2 + 9y^2 = t$$

is 5.

50. Find a number t such that the area inside the ellipse

$$2x^2 + 3y^2 = t$$

is 7.

PROBLEMS

51. Explain why a square yard contains 9 square feet.
52. Explain why a square foot contains 144 square inches.
53. Find a formula that gives the area of a square in terms of the length of the diagonal of the square.
54. Find a formula that gives the area of a square in terms of the perimeter.
55. Suppose a and b are positive numbers. Draw a figure of a square whose sides have length $a + b$. Partition this square into a square whose sides have length a, a square whose sides have length b, and two rectangles in a way that illustrates the identity

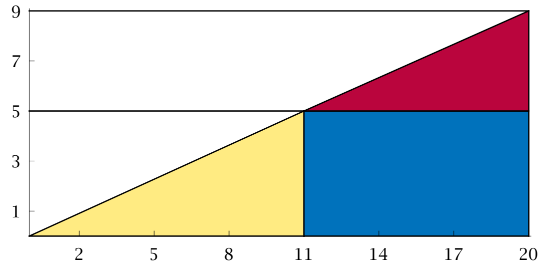
$$(a + b)^2 = a^2 + 2ab + b^2.$$

56. Find an example of a parallelogram whose area equals 10 and whose perimeter equals 16 (give the coordinates for all four vertices of your parallelogram).
57. Show that an equilateral triangle with sides of length r has area $\frac{\sqrt{3}}{4}r^2$.
58. Show that an equilateral triangle with area A has sides of length $\frac{2\sqrt{A}}{3^{1/4}}$.
59. Suppose $0 < a < b$. Show that the area of the region under the line $y = x$, above the x-axis, and between the lines $x = a$ and $x = b$ is $\frac{b^2-a^2}{2}$.
60. Show that the area inside a circle with circumference c is $\frac{c^2}{4\pi}$.
61. Find a formula that gives the area inside a circle in terms of the diameter of the circle.
62. In ancient China and Babylonia, the area inside a circle was said to be one-half the radius times the circumference. Show that this formula agrees with our formula for the area inside a circle.
63. Suppose a, b, and c are positive numbers. Show that the area inside the ellipse

$$ax^2 + by^2 = c$$

is $\pi\frac{c}{\sqrt{ab}}$.

64. Consider the following figure, which is drawn accurately to scale:



(a) Show that the right triangle whose vertices are $(0, 0)$, $(20, 0)$, and $(20, 9)$ has area 90.

(b) Show that the yellow right triangle has area 27.5.

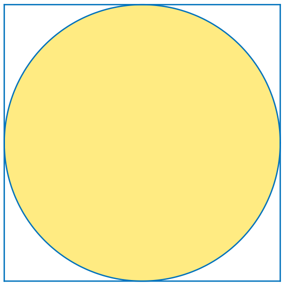
(c) Show that the blue rectangle has area 45.

(d) Show that the red right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect that parts (a) and (e) will have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.

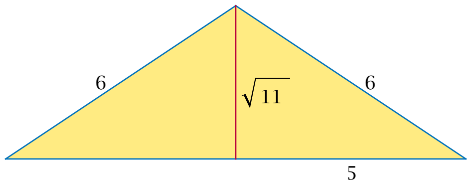
65. The figure below illustrates a circle of radius 1 enclosed within a square. By comparing areas, explain why this figure shows that $\pi < 4$.



WORKED-OUT SOLUTIONS *to Odd-numbered Exercises*

1. Find the area of a triangle that has two sides of length 6 and one side of length 10.

SOLUTION By the Pythagorean Theorem (see figure below), the height of this triangle equals $\sqrt{6^2 - 5^2}$, which equals $\sqrt{11}$.



A triangle that has two sides of length 6 *and one side of length* 10.

Thus the area of this triangle equals $5\sqrt{11}$.

3. (a) Find the distance from the point $(2,3)$ to the line containing the points $(-2,-1)$ and $(5,4)$.

   (b) Use the information from part (a) to find the area of the triangle whose vertices are $(2,3)$, $(-2,-1)$, and $(5,4)$.

SOLUTION

(a) To find the distance from the point $(2,3)$ to the line containing the points $(-2,-1)$ and $(5,4)$, we first find the equation of the line containing the points $(-2,-1)$ and $(5,4)$. The slope of this line equals

$$\frac{4-(-1)}{5-(-2)},$$

which equals $\frac{5}{7}$. Thus the equation of the line containing the points $(-2,-1)$ and $(5,4)$ is

$$\frac{y-4}{x-5} = \frac{5}{7},$$

which can be rewritten as

$$y = \tfrac{5}{7}x + \tfrac{3}{7}.$$

To find the distance from the point $(2,3)$ to the line containing the points $(-2,-1)$ and $(5,4)$, we want to find the equation of the line containing the point $(2,3)$ that is perpendicular to the line containing the points $(-2,-1)$ and $(5,4)$. The equation of this line is

$$\frac{y-3}{x-2} = -\frac{7}{5},$$

which can be rewritten as

$$y = -\tfrac{7}{5}x + \tfrac{29}{5}.$$

To find where this line intersects the line containing the points $(-2,-1)$ and $(5,4)$, we need to solve the equation

$$\tfrac{5}{7}x + \tfrac{3}{7} = -\tfrac{7}{5}x + \tfrac{29}{5}.$$

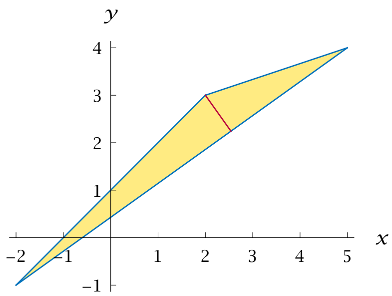
Simple algebra shows that the solution to this equation is $x = \frac{94}{37}$. Plugging this value of x into the equation of either line shows that $y = \frac{83}{37}$. Thus the two lines intersect at the point $(\frac{94}{37}, \frac{83}{37})$.

Thus the distance from the point $(2,3)$ to the line containing the points $(-2,-1)$ and $(5,4)$ is the distance from the point $(2,3)$ to the point $(\frac{94}{37}, \frac{83}{37})$. This distance equals

$$\sqrt{(2-\tfrac{94}{37})^2 + (3-\tfrac{83}{37})^2},$$

which equals $\sqrt{\frac{32}{37}}$, which equals $4\sqrt{\frac{2}{37}}$.

(b) We will consider the line segment connecting the points $(-2,-1)$, and $(5,4)$ to be the base of this triangle. In part (a), we found that the height of this triangle equals $4\sqrt{\frac{2}{37}}$.



The triangle with vertices $(2,3)$, $(-2,-1)$, *and* $(5,4)$, *with a line segment showing its height.*

The base of the triangle is the distance between the points $(-2,-1)$ and $(5,4)$. This distance equals $\sqrt{74}$. Thus the area of the triangle (one-half the base times the height) equals

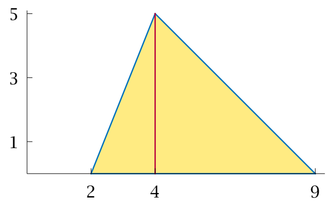
$$\tfrac{1}{2}\sqrt{74}\left(4\sqrt{\tfrac{2}{37}}\right),$$

which equals 4.

[There are easier ways to find the area of this triangle, but the technique used here gives you practice with several important concepts.]

5. Find the area of the triangle whose vertices are $(2,0)$, $(9,0)$, and $(4,5)$.

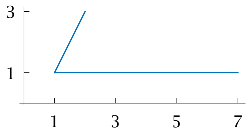
SOLUTION Choose the side connecting $(2,0)$ and $(9,0)$ as the base of this triangle. Thus the triangle below has base $9-2$, which equals 7.



The height of this triangle is the length of the red line shown here; this height equals the second coordinate of the vertex $(4,5)$. In other words, this triangle has height 5.

Thus this triangle has area $\frac{1}{2}\cdot 7\cdot 5$, which equals $\frac{35}{2}$.

7. Suppose $(2,3)$, $(1,1)$, and $(7,1)$ are three vertices of a parallelogram, two of whose sides are shown here.



(a) Find the fourth vertex of this parallelogram.

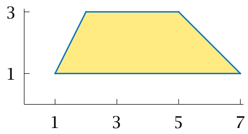
(b) Find the area of this parallelogram.

SOLUTION

(a) Consider the horizontal side of the parallelogram connecting the points $(1,1)$ and $(7,1)$. This side has length 6. Thus the opposite side, which connects the point $(2,3)$ and the fourth vertex, must also be horizontal and have length 6. Thus the second coordinate of the fourth vertex is the same as the second coordinate of $(2,3)$, and the first coordinate of the fourth vertex is obtained by adding 6 to the first coordinate of $(2,3)$. Hence the fourth vertex equals $(8,3)$.

(b) The base of this parallelogram is the length of the side connecting the points $(1,1)$ and $(7,1)$, which equals 6. The height of this parallelogram is the length of a vertical line segment connecting the two horizontal sides. Because one of the horizontal sides lies on the line $y=1$ and the other horizontal side lies on the line $y=3$, a vertical line segment connecting these two sides will have length 2. Thus the parallelogram has height 2. Because this parallelogram has base 6 and height 2, it has area 12.

9. Find the area of this trapezoid, whose vertices are $(1,1)$, $(7,1)$, $(5,3)$, and $(2,3)$.



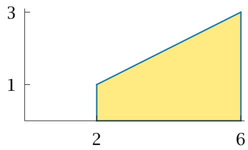
SOLUTION One base of this trapezoid is the length of the side connecting the points $(1,1)$ and $(7,1)$, which equals 6. The other base of this trapezoid is the length of the side connecting the points $(5,3)$ and $(2,3)$, which equals 3.

The height of this trapezoid is the length of a vertical line segment connecting the two horizontal sides. Because one of the horizontal sides lies on the line $y=1$ and the other horizontal side lies on the line $y=3$, a vertical line segment connecting these two sides will have length 2. Thus the trapezoid has height 2.

Because this trapezoid has bases 6 and 3 and has height 2, it has area $\frac{1}{2}(6+3)\cdot 2$, which equals 9.

11. Find the area of the region in the xy-plane under the line $y=\frac{x}{2}$, above the x-axis, and between the lines $x=2$ and $x=6$.

SOLUTION The line $x=2$ intersects the line $y=\frac{x}{2}$ at the point $(2,1)$. The line $x=6$ intersects the line $y=\frac{x}{2}$ at the point $(6,3)$.

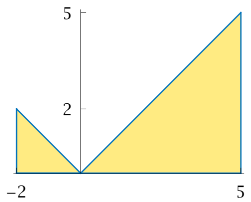


Thus the region in question is the trapezoid shown above. The parallel sides of this trapezoid (the two vertical sides) have lengths 1 and 3, and thus this trapezoid has bases 1 and 3. As can be seen from the figure above, this trapezoid has height 4. Thus the area of this trapezoid is $\frac{1}{2} \cdot (1+3) \cdot 4$, which equals 8.

13. Let $f(x) = |x|$. Find the area of the region in the xy-plane under the graph of f, above the x-axis, and between the lines $x = -2$ and $x = 5$.

SOLUTION

The region under consideration is the union of two triangles, as shown here.



One of the triangles has base 2 and height 2 and thus has area 2. The other triangle has base 5 and height 5 and thus has area $\frac{25}{2}$. Thus the area of the region under consideration equals $2 + \frac{25}{2}$, which equals $\frac{29}{2}$.

15. Find the area inside a circle with diameter 7.

SOLUTION A circle with diameter 7 has radius $\frac{7}{2}$. Thus the area inside this circle is $\pi\left(\frac{7}{2}\right)^2$, which equals $\frac{49\pi}{4}$.

17. Find the area inside a circle with circumference 5 feet.

SOLUTION Let r denote the radius of this circle in feet. Thus $2\pi r = 5$, which implies that $r = \frac{5}{2\pi}$. Thus the area inside this circle is $\pi\left(\frac{5}{2\pi}\right)^2$ square feet, which equals $\frac{25}{4\pi}$ square feet.

19. Find the area inside the circle whose equation is

$$x^2 - 6x + y^2 + 10y = 1.$$

SOLUTION To find the radius of the circle given by the equation above, we complete the square, as follows:

$$\begin{aligned} 1 &= x^2 - 6x + y^2 + 10y \\ &= (x-3)^2 - 9 + (y+5)^2 - 25 \\ &= (x-3)^2 + (y+5)^2 - 34. \end{aligned}$$

Adding 34 to both sides of this equation gives

$$(x-3)^2 + (y+5)^2 = 35.$$

Thus we see that this circle is centered at $(3, -5)$ (which is irrelevant for this exercise) and that it has radius $\sqrt{35}$. Thus the area inside this circle equals $\pi\sqrt{35}^2$, which equals 35π.

21. Find a number t such that the area inside the circle

$$3x^2 + 3y^2 = t$$

is 8.

SOLUTION Rewriting the equation above as

$$x^2 + y^2 = \left(\sqrt{\tfrac{t}{3}}\right)^2,$$

we see that this circle has radius $\sqrt{\frac{t}{3}}$. Thus the area inside this circle is $\pi\left(\sqrt{\frac{t}{3}}\right)^2$, which equals $\frac{\pi t}{3}$. We want this area to equal 8, which means we need to solve the equation $\frac{\pi t}{3} = 8$. Thus $t = \frac{24}{\pi}$.

Use the following information for Exercises 23–28:

23. What is the area of a DVD disk, not counting the hole?

SOLUTION The DVD disk has radius 6 cm (because the diameter is 12 cm). The area inside a circle with radius 6 cm is 36π square cm.

The area of the hole is $0.75^2\pi$ square cm, which is 0.5625π square cm.

Subtracting 0.5625π from 36π, we see that the DVD disk has area 34.4375π square cm, which is approximately 111.33 square cm.

25. A movie company is manufacturing DVD disks containing one of its movies. Part of the surface of each DVD disk will be made usable by coating with laser-sensitive material a circular ring whose inner radius is 2.25 cm from the center of the disk. What is the minimum outer radius of this circular ring if the movie requires 3100 megabytes of data storage?

SOLUTION Because 50.2 megabytes of data can be stored in each square cm of usable surface, the usable surface must have area at least $\frac{3100}{50.2}$ square cm, or about 61.753 square cm.

If the outer radius of the circular ring of usable area is r, then the usable area will be $\pi r^2 - 2.25^2\pi$. Thus we solve the equation

$$\pi r^2 - 2.25^2\pi = 61.753$$

for r, getting $r \approx 4.97$ cm.

27. Suppose a movie company wants to store data for extra features in a circular ring on a DVD disk. If the circular ring has outer radius 5.9 cm and 200 megabytes of data storage is needed, what is the maximum inner radius of the circular ring for the extra features?

SOLUTION Because 50.2 megabytes of data can be stored in each square cm of usable surface, the usable surface must have area at least $\frac{200}{50.2}$ square cm, or about 3.984 square cm.

If the inner radius of the circular ring of usable area is r, then the usable area will be $5.9^2\pi - \pi r^2$. Thus we solve the equation

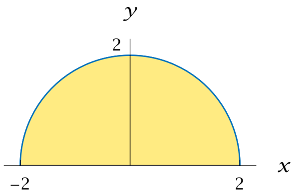
$$5.9^2\pi - \pi r^2 = 3.984$$

for r, getting $r \approx 5.79$ cm.

29. Find the area of the region in the xy-plane under the curve $y = \sqrt{4 - x^2}$ (with $-2 \le x \le 2$) and above the x-axis.

SOLUTION

Square both sides of the equation $y = \sqrt{4 - x^2}$ and then add x^2 to both sides.

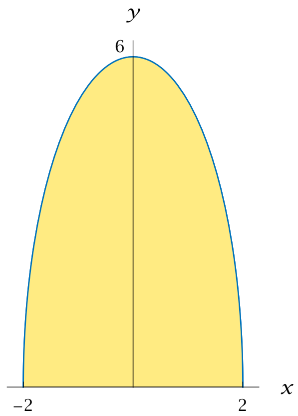


This gives the equation $x^2 + y^2 = 4$, which is the equation of a circle of radius 2 centered at the origin. However, the equation $y = \sqrt{4 - x^2}$ forces y to be nonnegative, and thus we have only the top half of the circle. Thus the region in question, which is shown above, has half the area inside a circle of radius 2. Hence the area of this region is $\frac{1}{2}\pi \cdot 2^2$, which equals 2π.

31. Using the answer from Exercise 29, find the area of the region in the xy-plane under the curve $y = 3\sqrt{4 - x^2}$ (with $-2 \le x \le 2$) and above the x-axis.

SOLUTION

The region in this exercise is obtained from the region in Exercise 29 by stretching vertically by a factor of 3. Thus by the Area Stretch Theorem, the area of this region is 3 times the area of the region in Exercise 29. Thus this region has area 6π.

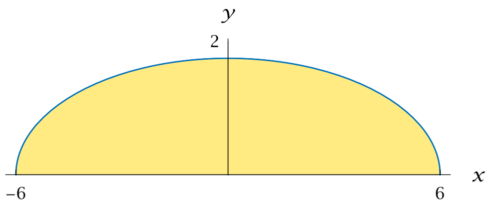


33. Using the answer from Exercise 29, find the area of the region in the xy-plane under the curve $y = \sqrt{4 - \frac{x^2}{9}}$ (with $-6 \le x \le 6$) and above the x-axis.

SOLUTION Define a function f with domain the interval $[-2, 2]$ by $f(x) = \sqrt{4 - x^2}$. Define a function h with domain the interval $[-6, 6]$ by $h(x) = f(\frac{x}{3})$. Thus

$$h(x) = f(\tfrac{x}{3}) = \sqrt{4 - (\tfrac{x}{3})^2} = \sqrt{4 - \tfrac{x^2}{9}}.$$

Hence the graph of h is obtained by horizontally stretching the graph of f by a factor of 3 (see Section 3.2). Thus the region in this exercise is obtained from the region in Exercise 29 by stretching horizontally by a factor of 3.



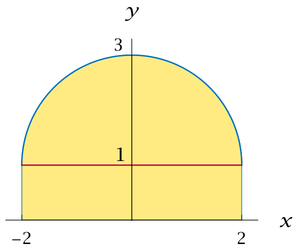
Thus by the Area Stretch Theorem, this region has area 6π.

35. Find the area of the region in the xy-plane under the curve

$$y = 1 + \sqrt{4 - x^2},$$

above the x-axis, and between the lines $x = -2$ and $x = 2$.

SOLUTION The curve $y = 1 + \sqrt{4 - x^2}$ is obtained by shifting the curve $y = \sqrt{4 - x^2}$ up 1 unit.



Thus we have the region above, which should be compared to the region shown in the solution to Exercise 29.

To find the area of this region, we break it into two parts. One part consists of the rectangle shown above that has base 4 and height 1 (and thus has area 4); the other part is obtained by shifting the region in Exercise 29 up 1 unit (and thus has area 2π, which is the area of the region in Exercise 29). Adding together the areas of these two parts, we conclude that the region shown above has area $4 + 2\pi$.

In Exercises 37–44, find the area inside the ellipse in the xy-plane determined by the given equation.

37. $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$

SOLUTION Rewrite the equation of this ellipse as

$$\frac{x^2}{\sqrt{7}^2} + \frac{y^2}{4^2} = 1.$$

Thus the area inside this ellipse is $4\sqrt{7}\pi$.

39. $2x^2 + 3y^2 = 1$

SOLUTION Rewrite the equation of this ellipse in the form given by the area formula, as follows:

$$\begin{aligned} 1 &= 2x^2 + 3y^2 \\ &= \frac{x^2}{\frac{1}{2}} + \frac{y^2}{\frac{1}{3}} \\ &= \frac{x^2}{\sqrt{\frac{1}{2}}^2} + \frac{y^2}{\sqrt{\frac{1}{3}}^2}. \end{aligned}$$

Thus the area inside the ellipse is $\pi \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{3}}$, which equals $\frac{\pi}{\sqrt{6}}$. Multiplying numerator and denominator by $\sqrt{6}$, we see that we could also express this area as $\frac{\sqrt{6}\pi}{6}$.

41. $3x^2 + 2y^2 = 7$

SOLUTION To put the equation of the ellipse in the form given by the area formula, begin by dividing both sides by 7, and then force the equation into the desired form, as follows:

$$\begin{aligned} 1 &= \tfrac{3}{7}x^2 + \tfrac{2}{7}y^2 \\ &= \frac{x^2}{\frac{7}{3}} + \frac{y^2}{\frac{7}{2}} \\ &= \frac{x^2}{\sqrt{\frac{7}{3}}^2} + \frac{y^2}{\sqrt{\frac{7}{2}}^2}. \end{aligned}$$

Thus the area inside the ellipse is $\pi \cdot \sqrt{\frac{7}{3}} \cdot \sqrt{\frac{7}{2}}$, which equals $\frac{7\pi}{\sqrt{6}}$. Multiplying numerator and denominator by $\sqrt{6}$, we see that we could also express this area as $\frac{7\sqrt{6}\pi}{6}$.

43. $3x^2 + 4x + 2y^2 + 3y = 2$

SOLUTION To put the equation of this ellipse in a standard form, we complete the square, as follows:

$$\begin{aligned} 2 &= 3x^2 + 4x + 2y^2 + 3y \\ &= 3[x^2 + \tfrac{4}{3}x] + 2[y^2 + \tfrac{3}{2}y] \\ &= 3[(x + \tfrac{2}{3})^2 - \tfrac{4}{9}] + 2[(y + \tfrac{3}{4})^2 - \tfrac{9}{16}] \\ &= 3(x + \tfrac{2}{3})^2 - \tfrac{4}{3} + 2(y + \tfrac{3}{4})^2 - \tfrac{9}{8} \\ &= 3(x + \tfrac{2}{3})^2 + 2(y + \tfrac{3}{4})^2 - \tfrac{59}{24}. \end{aligned}$$

Adding $\frac{59}{24}$ to both sides of this equation gives

$$3(x + \tfrac{2}{3})^2 + 2(y + \tfrac{3}{4})^2 = \tfrac{107}{24}.$$

Now multiplying both sides of this equation by $\frac{24}{107}$ gives

$$\tfrac{72}{107}\left(x+\tfrac{2}{3}\right)^2+\tfrac{48}{107}\left(y+\tfrac{3}{4}\right)^2=1.$$

We rewrite this equation in the form

$$\frac{\left(x+\frac{2}{3}\right)^2}{\left(\frac{\sqrt{107}}{\sqrt{72}}\right)^2}+\frac{\left(y+\frac{3}{4}\right)^2}{\left(\frac{\sqrt{107}}{\sqrt{48}}\right)^2}=1.$$

Thus the area inside this ellipse is

$$\pi\frac{\sqrt{107}}{\sqrt{72}}\frac{\sqrt{107}}{\sqrt{48}}.$$

Because $\sqrt{72}\sqrt{48}=\sqrt{36\cdot2}\sqrt{16\cdot3}=6\sqrt{2}\cdot4\sqrt{3}$, this equals

$$\pi\frac{107}{24\sqrt{6}}.$$

Multiplying numerator and denominator by $\sqrt{6}$ also allows us to express this area as

$$\pi\frac{107\sqrt{6}}{144}.$$

45. Find a positive number c such that the area inside the ellipse

$$2x^2+cy^2=5$$

is 3.

SOLUTION To put the equation of the ellipse in the form given by the area formula, begin by dividing both sides by 5, and then force the equation into the desired form, as follows:

$$\begin{aligned}1&=\tfrac{2}{5}x^2+\tfrac{c}{5}y^2\\&=\frac{x^2}{\frac{5}{2}}+\frac{y^2}{\frac{5}{c}}\\&=\frac{x^2}{\sqrt{\frac{5}{2}}^2}+\frac{y^2}{\sqrt{\frac{5}{c}}^2}.\end{aligned}$$

Thus the area inside the ellipse is $\pi\cdot\sqrt{\frac{5}{2}}\cdot\sqrt{\frac{5}{c}}$, which equals $\frac{5\pi}{\sqrt{2c}}$. We want this area to equal 3, so we must solve the equation $\frac{5\pi}{\sqrt{2c}}=3$. Squaring both sides and then solving for c gives $c=\frac{25\pi^2}{18}$.

47. Find numbers a and b such that $a>b$, $a+b=15$, and the area inside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is 36π.

SOLUTION The area inside the ellipse is πab. Thus we need to solve the simultaneous equations

$$a+b=15\quad\text{and}\quad ab=36.$$

The first equation can be rewritten as $b=15-a$, and this value for b can then be substituted into the second equation, giving the equation

$$a(15-a)=36.$$

This is equivalent to the equation $a^2-15a+36=0$, whose solutions (which can be found through either factoring or the quadratic formula) are $a=3$ and $a=12$. Choosing $a=3$ gives $b=15-a=12$, which violates the condition that $a>b$. Choosing $a=12$ gives $b=15-12=3$. Thus the only solution to this exercise is $a=12$, $b=3$.

49. Find a number t such that the area inside the ellipse

$$4x^2+9y^2=t$$

is 5.

SOLUTION Dividing the equation above by t, we have

$$\begin{aligned}1&=\tfrac{4}{t}x^2+\tfrac{9}{t}y^2\\&=\frac{x^2}{\frac{t}{4}}+\frac{y^2}{\frac{t}{9}}\\&=\frac{x^2}{\left(\frac{\sqrt{t}}{2}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{t}}{3}\right)^2}.\end{aligned}$$

Thus the area inside this ellipse is

$$\pi\frac{\sqrt{t}}{2}\cdot\frac{\sqrt{t}}{3},$$

which equals $\frac{\pi t}{6}$. Hence we want $\frac{\pi t}{6}=5$, which means that $t=\frac{30}{\pi}$.

CHAPTER SUMMARY

To check that you have mastered the most important concepts and skills covered in this chapter, make sure that you can do each item in the following list:

- Locate points on the coordinate plane.
- Compute the distance between two points.
- Find the equation of a line given its slope and a point on it.
- Find the equation of a line given two points on it.
- Find the equation of a line parallel to a given line and containing a given point.
- Find the equation of a line perpendicular to a given line and containing a given point.
- Find the midpoint of a line segment.
- Use the completing-the-square technique with quadratic expressions.
- Solve quadratic equations.
- Find the equation of a circle, given its center and radius.
- Find the vertex of a parabola.
- Compute the area of triangles and trapezoids.
- Compute the area inside a circle or ellipse.
- Explain how area changes when stretching either horizontally or vertically or both.

To review a chapter, go through the list above to find items that you do not know how to do, then reread the material in the chapter about those items. Then try to answer the chapter review questions below without looking back at the chapter.

CHAPTER REVIEW QUESTIONS

1. Find the distance between the points $(5, -6)$ and $(-2, -4)$.
2. Find two points, one on the horizontal axis and one on the vertical axis, such that the distance between these two points equals 21.
3. Find the perimeter of the parallelogram whose vertices are $(2, 1)$, $(7, 1)$, $(10, 3)$, and $(5, 3)$.
4. Find the perimeter of the triangle whose vertices are $(1, 2)$, $(6, 2)$, and $(7, 5)$.
5. Find the perimeter of the trapezoid whose vertices are $(2, 3)$, $(8, 3)$, $(9, 5)$, and $(-1, 5)$.
6. Explain how to find the slope of a line if given the coordinates of two points on the line.
7. Given the slopes of two lines, how can you determine whether or not the lines are parallel?
8. Given the slopes of two lines, how can you determine whether or not the lines are perpendicular?
9. Find a number t such that the line containing the points $(3, -5)$ and $(-4, t)$ has slope -6.
10. Find the equation of the line in the xy-plane that has slope -4 and contains the point $(3, -7)$.
11. Find the equation of the line in the xy-plane that contains the points $(-6, 1)$ and $(-1, -8)$.
12. Find the equation of the line in the xy-plane that is perpendicular to the line $y = 6x - 7$ and that contains the point $(-2, 9)$.
13. Find a line segment that is not parallel to either of the coordinate axes and that has $(-3, 5)$ as its midpoint.
14. Find the vertex of the graph of the equation
$$y = 5x^2 + 2x + 3.$$
15. Give an example of numbers a, b, and c such that the graph of
$$y = ax^2 + bx + c$$
has its vertex at the point $(-4, 7)$.

16. Find a number c such that the equation

$$x^2 + cx + 3 = 0$$

has exactly one solution.

17. Find a number x such that

$$\frac{x+1}{x-2} = 3x.$$

18. Find the equation of the circle in the xy-plane centered at $(-4, 3)$ that has radius 6.

19. Find the center, radius, and circumference of the circle in the xy-plane described by

$$x^2 - 8x + y^2 + 10y = 2.$$

Also, find the area inside this circle.

20. Find the area of a triangle that has two sides of length 8 and one side of length 3.

21. Find the area of the parallelogram whose vertices are $(2, 1)$, $(7, 1)$, $(10, 3)$, and $(5, 3)$.

22. Find the area of the triangle whose vertices are $(1, 2)$, $(6, 2)$, and $(7, 5)$.

23. Find the area of the trapezoid whose vertices are $(2, 3)$, $(8, 3)$, $(9, 5)$, and $(-1, 5)$.

24. Find a number t such that the area inside the circle

$$x^2 + 6x + y^2 - 8y = t$$

is 11.

25. Find the area inside the ellipse

$$3x^2 + 2y^2 = 5.$$

26. Suppose a newly discovered planet is orbiting a far-away star and that units and a coordinate system have been chosen so that planet's orbit is described by the equation

$$\frac{x^2}{29} + \frac{y^2}{20} = 1.$$

What are the two possible locations of the star?